HW: Let $T: H \to H$ be a self adjoint operator.
For $\lambda \in \mathbb{C}$ define
$$H_\lambda = \{v \in H \mid Tv = \lambda v\}.$$
If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

Proof Assume $H_\lambda \neq 0$.
Then there exists $v \in H$ such that $v \neq 0$ and $Tv = \lambda v$.
Since T is self adjoint and $\langle\ \rangle$ is Hermitian then
$$\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}.$$
So $\langle Tv, v\rangle \in \mathbb{R}$.
So $\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle \in \mathbb{R}$.
Since $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$ and $\langle v, v\rangle \neq 0$ then $\lambda \in \mathbb{R}$.

HW Let $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$. Then $H_\lambda \perp H_\mu$.

Proof: To show: $H_\lambda \perp H_\mu$
To show: If $x \in H_\lambda$ and $y \in H_\mu$ then $\langle x, y\rangle = 0$.
Assume $x \in H_\lambda$ and $y \in H_\mu$.
To show: $\langle x, y\rangle = 0$.
$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle$$
$$= \langle x, \mu y\rangle = \bar{\mu}\langle x, y\rangle = \mu\langle x, y\rangle.$$
So $(\lambda - \mu)\langle x, y\rangle = 0$.
Since $\lambda \neq \mu$ then $\langle x, y\rangle = 0$. //

HW: If T is compact and $\lambda \neq 0$ then H_λ is finite dimensional.

Proof To show: If H_λ ($\lambda \neq 0$) is infinite dimensional then T is not compact.

Assume $\lambda \neq 0$ and H_λ is infinite dimensional.

Let $(e_1, e_2, \ldots)$ be an orthonormal sequence in H_λ.

If $m \neq n$ then

$$\|Te_m - Te_n\| = \|\lambda e_m - \lambda e_n\| = |\lambda| \cdot \|e_m - e_n\| = |\lambda| \cdot \sqrt{2}.$$

So no subsequence of $(Te_1, Te_2, \ldots)$ is Cauchy.

So $(Te_1, Te_2, \ldots)$ does not have a convergent subsequence

So T is not compact. //

<u>Fredholm's Theorem</u> Let H be a Hilbert space and let $T: H \to H$ be a bounded compact self adjoint operator. Let $\lambda \in \mathbb{C}$.
Then

$\lambda - T$ is injective if and only if $\lambda - T$ is bijective.

<u>Proof</u> $\Leftarrow$ By the definition of bijective, if $\lambda - T$ is bijective then $\lambda - T$ is injective.

$\Rightarrow$ Assume that $\lambda - T$ is a linear operator and $\lambda - T$ is injective.

To show: If T is compact then $\lambda - T$ is bijective.

To show: If T is compact then $\lambda - T$ is surjective.

To show: If $\mathrm{im}(\lambda - T) \neq H$ then T is not compact.

Assume $(\lambda - T)(H) \subsetneq H$.

To show: T is not compact.

Construct

$$\cdots \subsetneq (\lambda - T)^2(H) \subsetneq (\lambda - T)(H) \subsetneq H$$

Let $e_n \in (\lambda - T)^n(H) \cap (\lambda - T)^{n+1}(H)^\perp$ with $\|e_n\| = 1$.

~~Then~~ If $m, n \in \mathbb{Z}_{>0}$ and $m < n$ then

$$\|Te_m - Te_n\| \geq \|\lambda e_m\| = |\lambda|$$

So $(Te_1, Te_2, \ldots)$ is not Cauchy.

So $(Te_1, Te_2, \ldots)$ does not have a convergent subsequence. So T is not compact.

<u>Proposition</u> Assume T is compact.

If $(\lambda_1, \lambda_2, \dots)$ is a sequence of distinct eigenvalues in $\sigma_p(T)$ then

$$\lim_{n\to\infty} \lambda_n = 0$$

<u>Proof</u> (Sketch)

Let $(\lambda_1, \lambda_2, \dots)$ be a sequence of distinct eigenvalues in $\sigma_p(T)$.

Assume $\lim_{n\to\infty} \lambda_n \neq 0$

To show: T is not compact.

Let $u_n \in H_{\lambda_n}$ with $\|u_n\| = 1$ and let

$W_n = \mathrm{span}\{u_1, u_2, \dots\}$. so that

$0 \subsetneq W_1 \subsetneq W_2 \subseteq \dots$

Let $e_n \in W_n \cap W_{n-1}^{\perp}$ with $\|e_n\| = 1$.

Then $\|Te_n - Te_m\| = \cdots \geq \cdots = |\lambda_n\|$

Since $\lim_{n\to\infty} \lambda_n \neq 0$ then $(Te_1, Te_2, \dots)$ is not Cauchy

So $(Te_1, Te_2, \dots)$ does not have a convergent subsequence.

So T is not compact.

Theorem 3 Let $(H, \langle , \rangle)$ be a Hilbert space and let $T: H \to H$ be a nonzero compact self adjoint linear operator. Let

$(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \|T\|.$$

Let y be a cluster point of $(Tu_1, Tu_2, \ldots)$
then $\frac{y}{\|T\|}$ is a cluster point of $(u_1, u_2, \ldots)$
$\|y\| = \|T\|$, $\quad \frac{|\langle Ty, y\rangle|}{\|y\|^2} = \|T\|$ and $Ty = \|T\|y$.

Proof Let $\lambda = \|T\|$.

Since y is a cluster point of $(Tu_1, Tu_2, \ldots)$ there exists a subsequence $(Tu_{n_1}, Tu_{n_2}, \ldots)$ such that

$$\lim_{k\to\infty} Tu_{n_k} = y.$$

(a) To show: $\lim_{k\to\infty} u_{n_k} = \frac{y}{\|T\|}$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>N}$ then $\|y - \lambda u_{n_k}\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $N_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>N_1}$ then $\|y - Tu_{n_k}\| < \frac{\varepsilon}{10}$

Using that $\lim_{k\to\infty} (T-\lambda)u_{n_k} = 0$,

let $N_2 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>N_2}$ then $\|Tu_{n_k} - \lambda u_{n_k}\| < \frac{\varepsilon}{10}$

19.08.2022
MHS Lect 12 4

Let $N = \max\{N_1, N_2\}$.

then

$$\|y - \lambda u_{n_k}\| \leq \|y - Tu_{n_k}\| + \|Tu_{n_k} - \lambda u_{n_k}\|$$

$$\leq \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

So $\lim_{k\to\infty} \lambda u_{n_k} = y$.

(b) Since $\| \, \| : H \to \mathbb{R}_{\geq 0}$ is continuous then

$$\|y\| = \|\lim_{k\to\infty} \lambda u_{n_k}\| = \lim_{k\to\infty} \|\lambda u_{n_k}\|$$

$$= \lim_{k\to\infty} |\lambda| \, \|u_{n_k}\| = |\lambda| = \|T\|.$$

(c) Since $\| \, \| : H \to \mathbb{R}_{\geq 0}$ is continuous and T is continuous then

$$\|Ty - \lambda y\| = \|T(\lim_{k\to\infty} \lambda u_{n_k}) - \lambda y\|$$

$$= \lim_{k\to\infty} \lambda \|Tu_{n_k} - y\| = 0.$$

Since $\|y\| = |\lambda| = \|T\|$ and T is not the 0 operator then $y \neq 0$ and

$$Ty = \lambda y \text{ and } \langle Ty, y\rangle = \langle \lambda y, y\rangle = \lambda \|y\|^2.$$

//