Limits and Continuity

Definitions Let (X, d_X) be a metric space.
Let $a \in X$.

ε-ball at a

$$B_\varepsilon(a) = \{x \in X \mid d_X(x,a) < \varepsilon\}.$$

Neighborhoods of a

$$\mathcal{N}(a) = \left\{ N \subseteq X \;\middle|\; \begin{array}{l} \text{there exists } \varepsilon \in \mathbb{R}_{>0} \\ \text{such that } B_\varepsilon(a) \subseteq N \end{array} \right\}$$

Open sets in X

$$\mathcal{T}_X = \{U \subseteq X \mid \text{if } a \in U \text{ then } U \in \mathcal{N}(a)\}.$$

Closed sets in X

$$\mathcal{C}_X = \{C \subseteq X \mid C^c \in \mathcal{T}_X\}$$

where $C^c = \{x \in X \mid x \notin C\}$.

First kind of limits

Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $f: X \to Y$ be a function, $a \in X$ and $y \in Y$.

(a) $\lim_{x \to a} f(x) = y$ if f satisfies:

if $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that

if $x \in X$ and $d_X(x, a) < \delta$ then $d_Y(f(x), y) < \varepsilon$

(b) $\lim_{x \to a} f(x) = y$ if f satisfies:

If $B_\varepsilon(y)$ is an ε-ball at y then there exists $B_\delta(a)$ such that

$$B_\varepsilon(y) \supseteq f(B_\delta(a))$$

(c) $\lim_{x \to a} f(x) = y$ if f satisfies:

If $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that

$$N \supseteq f(P).$$

Let (X, d_X) and (Y, d_Y) be metric spaces.

Let $f: X \to Y$ be a function.

Let $a \in X$ and $y \in Y$.

<u>Second kind of limits</u>

(a) $\lim_{\substack{x \to a \\ x \neq a}} f(x) = f(y)$ if f satisfies:

If $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $0 < d_X(x, a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

(b) $\lim_{\substack{x \to a \\ x \neq a}} f(x) = f(y)$ if f satisfies:

if $B_\varepsilon(y)$ is an ε-ball at y then there exists $B_\delta(a)$ such that

$$B_\varepsilon(y) \supseteq f(B_\delta(a) - \{a\}).$$

(c) $\lim_{\substack{x \to a \\ x \neq a}} f(x) = y$ if f satisfies:

If $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that

$$N \supseteq f(P - \{a\}).$$

Third kind of limits

Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Let $z \in X$.

(1) $\lim_{n\to\infty} x_n = z$ if $(x_1, x_2, \ldots)$ satisfies:

If $\varepsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, z) < \varepsilon$.

(2) $\lim_{n\to\infty} x_n = z$ if $(x_1, x_2, \ldots)$ satisfies:

If $\varepsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

$$B_\varepsilon(z) \supseteq \{x_\ell, x_{\ell+1}, \ldots\}.$$

(3) $\lim_{n\to\infty} x_n = z$ if $(x_1, x_2, \ldots)$ satisfies:

If $P \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

$$P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}.$$

Continuous at a

Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $f: X \to Y$ be a function. Let $a \in X$.

(a) The function f is continuous at a if f satisfies

$$\lim_{\substack{x \to a \\ x \neq a}} f(x) = f(a)$$

(b) The function f is continuous at a if f satisfies:

if $\varepsilon \in \mathbb{E}$ ~~then~~ then there exists $\delta \in \mathbb{E}$ such that

if $x \in X$ and $0 < d_X(x,a) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$

(c) The function f is continuous at a if f satisfies:

if $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

(d) The function f is continuous at a if f satisfies:

if $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$

(e) The function f is continuous at a if f satisfies:

if $U \in \mathcal{T}_X$ and $a \in U$ then $f^{-1}(U) \in \mathcal{T}_Y$.

Definitions of continuous

Let $f: X \to Y$ be a function.

(a) The function f is <u>continuous</u> if f satisfies:

if $a \in X$ then f is continuous at a.

(b) The function f is <u>continuous</u> if f satisfies

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

(c) The function f is <u>continuous</u> if f satisfies

if $C \in \mathcal{C}_Y$ then $f^{-1}(C) \in \mathcal{C}_X$.