Generating topologies

Let $\mathcal{B}$ be a collection of subsets of X.

The topology generated by $\mathcal{B}$ is the collection $\mathcal{T}$ of subsets of X such that

(a) $\mathcal{T}$ is a topology on X and $\mathcal{T} \supseteq \mathcal{B}$.

(b) If $\mathcal{T}'$ is a topology on X and $\mathcal{T}' \ni \mathcal{B}$ then $\mathcal{T}' \ni \mathcal{T}$.

Remark: To show that the "smallest topology containing $\mathcal{B}$" exists prove that

(a) The set $\mathcal{P}$ of all subsets of X is a topology on X (the discrete topology).

(b) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.

Then the topology on X generated by $\mathcal{B}$ is the intersection of all topologies on X which contain $\mathcal{B}$.

Proposition Let X be a set and let $\mathcal{B}$ be a collection of subsets of X which satisfy

(a) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$

(b) $\bigcup_{B \in \mathcal{B}} B = X$.

then the topology generated by $\mathcal{B}$ is the set of unions of elements of $\mathcal{B}$,

$$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \text{there exists } \mathcal{S} \subseteq \mathcal{B} \\ \text{with } U = \bigcup_{B \in \mathcal{S}} B \end{array} \right\}$$

Proposition Let (X,d) be a metric space and let

$$\mathcal{B} = \{ B_\varepsilon(x) \mid \varepsilon \in \mathbb{E} \text{ and } x \in X \}.$$

then $\mathcal{B}$ satisfies (a) and (b) of the previous proposition and the metric space topology is the collection of unions of elements of $\mathcal{B}$ (unions of ε-balls).

<u>Hausdorff spaces: Uniqueness of limits</u> MHS Lect 17

A <u>Hausdorff</u> <u>topological space</u> is a topological space $(X, \mathcal{T}_X)$ such that

if $z_1, z_2 \in X$ and $z_1 \neq z_2$ then there exist

$N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$ with $N_1 \cap N_2 = \phi$.

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space. Let $(a_1, a_2, \ldots)$ be a sequence in X. Then $(a_1, a_2, \ldots)$ has at most one limit point in X.

<u>Proof</u> Let $(a_1, a_2, \ldots)$ be a sequence in X.

Assume $z_1, z_2 \in X$ with

$$\lim_{n \to \infty} a_n = z_1 \quad \text{and} \quad \lim_{n \to \infty} a_n = z_2.$$

Let $N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$.

Then there exists $\ell_1 \in \mathbb{Z}_{>0}$ with

$$N_1 \supseteq \{a_{\ell_1}, a_{\ell_1+1}, \ldots\}$$

and there exists $\ell_2 \in \mathbb{Z}_{>0}$ with

$$N_2 \supseteq \{a_{\ell_2}, a_{\ell_2+1}, \ldots\}.$$

Let $\ell = \max\{\ell_1, \ell_2\}$.

Then $a_z \in N_1 \cap N_2$ and $N_1 \cap N_2 \neq \emptyset$.

So $z_1 = z_2$, since $(X, \mathcal{T}_X)$ is Hausdorff. //

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ be a topological space. Let

$$\begin{aligned} \Delta : X &\to X \times X \\ x &\mapsto (x, x) \end{aligned} \quad \text{so that } \Delta(X) = \{(x,x) \mid x \in X\}.$$

Then $(X, \mathcal{T}_X)$ is Hausdorff if and only if

$$\Delta(X) \text{ is closed in } X \times X.$$

What is the topology on $X \times X$?

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

The <u>product topology</u> on $X \times Y$ is the topology generated by

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}.$$

Neighborhoods for the product topology:

$$N(x,y) = \left\{ N \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{there exist } U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \\ \text{with } x \in U \text{ and } y \in V \\ \text{and } U \times V \subseteq N \end{array} \right\}$$

<u>Proof</u> $\Rightarrow$: Assume $(X, \mathcal{T}_X)$ is Hausdorff.

To show: $\Delta(X)$ is closed in $X \times X$.

To show: If $(z_1, z_2) \in X \times X$ is a close point to $\Delta(X)$ then $(z_1, z_2) \in \Delta(X)$.

Assume (z_1, z_2) is a close point to $\Delta(X)$.

To show: $z_1 = z_2$.

~~Let $N \in \mathcal{N}(z_1, z_2)$~~.

Let $U \in \mathcal{N}(z_1)$ and $V \in \mathcal{N}(z_2)$. ~~with $U \times V \subseteq N$~~.

Since $U \times V \in \mathcal{N}(z_1, z_2)$ then

$$(U \times V) \cap \Delta(X) \neq \emptyset.$$

So there exists $z \in X$ with $(z, z) \in U \times V$.

So $U \cap V \neq \emptyset$.

Since $(X, \mathcal{T}_X)$ is Hausdorff then $z_1 = z_2$.

So $(z_1, z_2) = (z_1, z_1) \in \Delta(X)$.

So $\Delta(X)$ is closed in $X \times X$.

$\Leftarrow$: Assume $\Delta(X)$ is closed in $X \times X$.

To show: $(X, \mathcal{T}_X)$ is Hausdorff.

To show: If $x_1, x_2 \in X$ and $x_1 \neq x_2$ then there exist $N_1 \in \mathcal{N}(x_1)$ and $N_2 \in \mathcal{N}(x_2)$ with $N_1 \cap N_2 = \emptyset$.

Assume $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Then $(x_1, x_2) \notin \Delta(X)$.

Since $\Delta(X)$ is closed then (x_1, x_2) is not a close point to $\Delta(X)$.

So there exists $N \in \mathcal{N}(x_1, x_2)$ such that $N \cap \Delta(X) = \emptyset$.

Since $N \in \mathcal{N}(x_1, x_2)$ there exist $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_X$ with $x_1 \in U$ and $x_2 \in V$ and $U \times V \subseteq N$.

~~Since~~ Let $N_1 = U$ and $N_2 = V$.

Then $N_1 \in \mathcal{N}(x_1)$ and $N_2 \in \mathcal{N}(x_2)$.

Since $(U \times V) \cap \Delta(X) = \emptyset$ then $U \cap V = \emptyset$.

So $N_1 \cap N_2 = \emptyset$.

So $(X, \mathcal{T}_X)$ is Hausdorff.