New topological spaces from old

Subspaces Let $(X, \mathcal{T}_X)$ be a topological space. Let S be a subset of X. Then

$$\mathcal{T}_S = \{S \cap U \mid U \in \mathcal{T}_X\}$$

is a topology on S, the subspace topology.

The subspace topology is the minimal topology on S such that

$$\begin{matrix} S & \longrightarrow & X \\ s & \longmapsto & s \end{matrix}$$

is a continuous function.

The quotient topology

Let $(X, \mathcal{T}_X)$ be a topological space.
Let $\sim$ be an equivalence relation on X.
An equivalence relation on X is a subset R of $X \times X$,

$$R = \{(s,t) \mid s \sim t\},$$

satisfying

(a) if $s \in X$ then $s \sim s$,
(b) if $s, t \in X$ and $s \sim t$ then $t \sim s$.
(c) if $s, t, u \in X$ and $s \sim t$ and $t \sim u$ then $s \sim u$.

Let $s \in X$. The <u>equivalence class of s</u> is

$$[s] = \{t \in X \mid t \sim s\}.$$

The <u>quotient of X by $\sim$</u> is the set

$$X/\!\sim\; = \{[s] \mid s \in X\}.$$

The <u>quotient topology</u> is the minimal topology on $X/\!\sim$ such that

$$\begin{aligned} X &\to X/\!\sim \\ s &\mapsto [s] \end{aligned}$$

is a continuous function.

<u>Products</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The <u>product topology</u> is the topology on $X \times Y$ generated by

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}.$$

The product topology is the minimal topology on $X \times Y$ such that

$$\begin{aligned} X \times Y &\to X \\ (x,y) &\mapsto x \end{aligned} \quad \text{and} \quad \begin{aligned} X \times Y &\to Y \\ (x,y) &\mapsto y \end{aligned}$$

are continuous functions.

<u>Universal property</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The <u>product of X and Y</u> is

(a) a topological space $(X \times Y, \mathcal{T}_{X \times Y})$
with continuous functions
$$X \times Y \xrightarrow{p_1} X \quad \text{and} \quad X \times Y \xrightarrow{p_2} Y$$

(b) If $(Z, \mathcal{T}_Z)$ is a topological space
with continuous functions
$$Z \xrightarrow{f_1} X \quad \text{and} \quad Z \xrightarrow{f_2} Y$$
then there exists a unique continuous function
$$g : Z \to X \times Y \quad \text{such that} \quad \begin{array}{l} f_1 = p_1 \circ g \\ f_2 = p_2 \circ g. \end{array}$$

$$\begin{tikzcd} & & X \\ Z \arrow[r, dashed, "g"'] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & X \times Y \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\ & & Y \end{tikzcd}$$

<u>Continuous functions</u>

Continuous functions are for comparing topological spaces

02.09.2022
MHS Lect 18

Categories

Categories have Objects and Morphisms

Morphisms are for comparing objects.

Examples:

Category	Objects	Morphisms
Sets	X	functions
Topological spaces	$(X, \mathcal{T}_X)$	continuous functions
Groups	$(G, \cdot)$	homomorphisms
Vector spaces	$(V,$ addition and scalar mult$)$	linear transformations

Point of "category theory": (universal property)

If you know the definition of product for one category then you know it for all categories.

Products of metric spaces?

Let (X, d_X) and (Y, d_Y) be metric spaces.

Some favourite metrics on $X \times Y$ are

(1) $d_1 : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

(2) $d_2 : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

(3) $d_\infty : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

(4) Let $p \in \mathbb{R}_{>1}$. Let $d_p : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_p((x_1, y_1), (x_2, y_2)) = \left(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p\right)^{1/p}.$$

Any of these might be called the product metric space.

Let

$\mathcal{T}_1$ be the metric space topology for $(X \times Y, d_1)$

$\mathcal{T}_2$ be the metric space topology for $(X \times Y, d_2)$

$\mathcal{T}_\infty$ be the metric space topology for $(X \times Y, d_\infty)$

$\mathcal{T}_p$ be the metric space topology for $(X \times Y, d_p)$

Let

$\mathcal{T}_X$ be the metric space topology for (X, d_X)

$\mathcal{T}_Y$ be the metric space topology for (Y, d_Y).

Let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$ coming from $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

Show that

$$\mathcal{T}_{X \times Y} = \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_\infty = \mathcal{T}_p.$$