07.09.2021
MHS Lect 20

Compactness – Existence of cluster points

Let (X, d_X) be a metric space. Let $A \subseteq X$.

The set A is <u>sequentially compact</u> if
every sequence in A has a cluster point in A.

The set A is <u>Cauchy compact</u> if
every Cauchy sequence in A has a cluster point in A.

The set A is <u>cover compact</u> if
every open cover of A has a finite subcover.

For $\varepsilon \in \mathbb{E}$ let $\mathcal{S}_\varepsilon = \{B_\varepsilon(a) \mid a \in A\}$.

The set A is <u>ball compact</u> if
every $\mathcal{S}_\varepsilon$ has a finite subcover.

<u>Theorem</u> The following are equivalent:

(a) A is cover compact
(b) A is sequentially compact
(c) A is ball compact and Cauchy compact.

$$\begin{tikzcd}
\text{cover compact} \arrow[d, Rightarrow] & \\
\text{sequentially compact} \arrow[r, Rightarrow] & \text{ball compact and Cauchy compact} \arrow[ul, Rightarrow]
\end{tikzcd}$$

<u>Cover compact $\Rightarrow$ sequentially compact</u>

<u>Proof</u>: To show: If A is not sequentially compact then A is not cover compact.

Assume A is not sequentially compact.

Let $(a_1, a_2, \ldots)$ be a sequence in A with no cluster point in A.

Then, if $z \in A$ there exists $V_z \in \mathcal{T}_X$ such that V_z contains only finitely many of the sequence $(a_1, a_2, \ldots)$ (i.e. z is not a cluster point).

To show: A is not cover compact.

To show: There exists an open cover $\mathcal{S}$ of A with no finite subcover.

For $x \in A$ let V_x be an open set of A with

$x \in V_x$ and $(a_1, a_2, \ldots) \cap V_x$ finite.

Then $\mathcal{S} = \{V_x \mid x \in A\}$ is an open cover of A.

To show: $\mathcal{S}$ does not contain a finite subcover of A.

Assume $\ell \in \mathbb{Z}_{>0}$ and $V_{x_1}, V_{x_2}, \ldots, V_{x_\ell} \in \mathcal{S}$.

For $j \in \{1, \ldots, \ell\}$
Let $k_j \in \mathbb{Z}_{>0}$ be such that

if $n \in \mathbb{Z}_{>k_j}$ then $a_n \notin V_{k_j}$.

Let $k = \max\{k_1, \ldots, k_\ell\}$.

Then $(a_{k+1}, a_{k+2}, \ldots) \cap (V_{k_1} \cup \cdots \cup V_{k_\ell}) = \phi$.

So $V_{k_1} \cup \cdots \cup V_{k_\ell}$ does not cover A.

So $\mathcal{S}$ has no finite subcover.

So A is not cover compact.

<u>Sequentially compact $\Rightarrow$ ball compact</u>

<u>Proof</u> To show: If A is not ball compact then A is not sequentially compact.

Assume A is not ball compact.

To show: There exists $(a_1, a_2, \ldots)$ in A with no cluster point in A.

Let $\varepsilon \in \mathbb{E}$ be such that $\mathcal{S}_\varepsilon = \{B_\varepsilon(a) \mid a \in A\}$ has no finite subcover.

Let $a_1 \in A$,

$a_2 \in A \cap B_{\frac{\varepsilon}{10}}(a_1)^c$,

$a_3 \in A \cap \left(B_{\frac{\varepsilon}{10}}(a_1) \cup B_{\frac{\varepsilon}{10}}(a_2)\right)^c$, ...

Then every $B_\varepsilon(a)$ contains at most one point of $(a_1, a_2, \ldots)$.

So $(a_1, a_2, \ldots)$ has no cluster point in A.

So A is not sequentially compact.

<u>Ball compact + Cauchy compact $\Rightarrow$ cover compact</u>

<u>Proof</u> Assume A is ball compact.

To show: If A is not cover compact then A is not Cauchy compact.

Assume A is not cover compact.

Let $\mathcal{S}$ be an open cover of A with no finite subcover of A.

Let $a_1^{(1)}, \ldots, a_{\ell_1}^{(1)} \in A$ be such that

$$B_{10^{-1}}(a_1^{(1)}) \cup \cdots \cup B_{10^{-1}}(a_{\ell}^{(1)}) \supseteq A.$$

Let $j_1 \in \{1, \ldots, \ell_1\}$ be such that

$A \cap B_{10^{-1}}(a_{j_1}^{(1)})$ is not finitely covered by $\mathcal{S}$.

Let $a_1^{(2)}, \ldots, a_{\ell_2}^{(2)} \in A$ be such that

$$B_{10^{-2}}(a_1^{(2)}) \cup \cdots \cup B_{10^{-2}}(a_{\ell_2}^{(2)}) \supseteq A \cap B_{10^{-1}}(a_{j_1}^{(1)}).$$

Let $j_2 \in \{1, \ldots, \ell_2\}$ such that

$$A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)}) \text{ is not finitely covered by } \mathcal{S}$$

Continuing this process produces a sequence $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$ in A, which is Cauchy

$$\left(\begin{array}{l} \text{If } r, s \geq k+1 \text{ then} \\ d(a_{j_r}^{(r)}, a_{j_s}^{(s)}) \leq d(a_{j_r}^{(r)}, a_{j_{k+1}}^{(k+1)}) + d(a_{j_{k+1}}^{(k+1)}, a_{j_s}^{(s)}) \\ \qquad \leq 10^{-(k+1)} + 10^{-(k+1)} < 10^{-k} \end{array} \right)$$

Let $z \in A$.

To show: z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$.

To show: There exist $\varepsilon \in \mathbb{R}$ and $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>\ell}$ then $d(a_{j_n}^{(n)}, z) > \varepsilon$.

Let $U \in \mathcal{S}$ such that $z \in U$.

Since U is open in X then there exists $k \in \mathbb{Z}_{>0}$ such that $B_{10^{-k}}(z) \subseteq U$.

Let $\varepsilon = 10^{-k}$ and let $\ell = k$.

07.09.2022
MHS Lect 20 ⑥

To show: If $n \in \mathbb{Z}_{\geq l}$ then $d(a_{j_n}^{(n)}, z) > \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq l}$.

Since $B_{10^{-n}}(a_{j_n}^{(n)}) \not\subseteq B_{10^{-k}}(z)$ then

there exists $y \in B_{10^{-n}}(a_{j_n}^{(n)})$ such that

$$d(y, z) > 10^{-k}.$$

So

$$d(a_{j_n}^{(n)}, z) \geq d(y, z) - d(a_{j_n}^{(n)}, y)$$

$$> 10^{-k} - 10^{-n} > 10^{-k} = \varepsilon$$

So z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$

So A is not Cauchy compact. //.