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MH5 Lect 25
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Number systems

$$\mathbb{Q}((t)) = \left\{ a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + \cdots \;\middle|\; \begin{matrix} \ell\in\mathbb{Z} \\ a_i\in\mathbb{Q} \end{matrix} \right\}$$

$$\cup|$$

$$\mathbb{Q}[[t]] = \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i\in\mathbb{Q} \}$$

$$\cup|$$

$$\mathbb{Q}[t] = \left\{ a_0 + a_1 t + a_2 t^2 + \cdots \;\middle|\; \begin{matrix} a_i\in\mathbb{Q} \text{ and all but} \\ \text{a finite number} \\ \text{of } a_i \text{ are } 0 \end{matrix} \right\}$$

Define a metric $d: \mathbb{Q}((t))\times\mathbb{Q}((t)) \to \mathbb{R}_{\geq 0}$ by

$$d(x,y) = 10^{-\mathrm{val}_t(y-x)},$$

where
$$\mathrm{val}_t(a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + \cdots) = \ell$$
if ℓ is minimal such that $a_\ell \neq 0$.

Examples:

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots$$

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots$$

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots$$

$$\tan t = \frac{\sin t}{\cos t}$$

p-adic numbers

For $p \in \mathbb{Z}_{>0}$ let $\mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p-1\}$.

$$\mathbb{Q}_p = \left\{ a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z} \\ a_i \in \mathbb{Z}/p\mathbb{Z} \end{matrix} \right\}$$

$$\cup\!|$$

$$\mathbb{Z}_p = \{ a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \mathbb{Z}/p\mathbb{Z} \}$$

$$\cup\!|$$

$$\mathbb{Z}_{\geq 0} = \left\{ a_0 + a_1 p + a_2 p^2 + \cdots \;\middle|\; \begin{matrix} a_i \in \mathbb{Z}/p\mathbb{Z} \text{ and all} \\ \text{but a finite number} \\ \text{of the } a_i \text{ are } 0 \end{matrix} \right\}$$

Define a metric $d : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ by

$$d(x, y) = 10^{-\mathrm{val}_p(y-x)}$$

where $\mathrm{val}_p(a_\ell p^\ell + a_{\ell+1}p^{\ell+1} + \cdots) = \ell$

if ℓ is minimal such that $a_\ell \neq 0$.

Real numbers

$$\mathbb{R}_{\geq 0} = \left\{ a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z} \\ a_i \in \mathbb{Z}/10\mathbb{Z} \end{matrix} \right\}$$

$$\cup\!|$$

$$\mathbb{Z}_{\geq 0} = \left\{ a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + \cdots + a_{-1}\left(\tfrac{1}{10}\right)^{-1} + a_0 \;\middle|\; \begin{matrix} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \mathbb{Z}/10\mathbb{Z} \end{matrix} \right\}$$

Define a metric $d: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$d(x,y) = 10^{-\mathrm{val}(y-x)}$$

where $\mathrm{val}\left(a_\ell \left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots\right) = \ell$

if ℓ is minimal such that $a_\ell \neq 0$.

The $\mathfrak{a}$-adic topology

Let G be an abelian group. Let

$G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be subgroups of G.

Define

$$\mathcal{N}(0) = \left\{ N \subseteq G \;\middle|\; \begin{matrix}\text{there exists } n \in \mathbb{Z}_{>0} \\ \text{with } N \supseteq G_n\end{matrix} \right\}$$

and $\mathcal{N}(g) = \{ g + N \mid N \in \mathcal{N}(0) \}$ for $g \in G$.

Define

$$\mathcal{E}_G = \left\{ D \subseteq G \times G \;\middle|\; \begin{matrix}\text{there exists } n \in \mathbb{Z}_{>0} \\ \text{with } D \supseteq B_n\end{matrix} \right\}$$

where

$$B_n = \{ (x,y) \in G \times G \mid y - x \in G_n \}$$

Define

$$\mathcal{T}_G = \left\{ U \subseteq G \;\middle|\; \begin{matrix}\text{if } g \in U \text{ then there exists} \\ n \in \mathbb{Z}_{>0} \text{ with } g + G_n \subseteq U\end{matrix} \right\}$$

<u>HW</u> Show that

(a) $\mathcal{E}_G$ is a uniformity on G

(b) $\mathcal{T}_G$ is the uniform space topology for $(G, \mathcal{E}_G)$

(c) The maps
$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g_1,g_2)\mapsto g_1+g_2 & & g \mapsto -g \end{array}$$
are continuous.

<u>Example</u> A is a ring, $\mathfrak{a}$ is an ideal of A.
$$G = A \quad \text{and} \quad G_n = \mathfrak{a}^n \quad \text{for } n\in \mathbb{Z}_{>0}.$$
Then $\mathcal{T}_G$ is the $\mathfrak{a}$-adic topology on A.

<u>Inverse limits</u>

A <u>coherent sequence</u> is a sequence $(\bar{a}_1, \bar{a}_2, \ldots)$ with
$$\bar{a}_n \in G/G_n \quad \text{and} \quad \pi_n(\bar{a}_{n+1}) = \bar{a}_n$$
where
$$\begin{array}{rccc} \pi_n: & G/G_{n+1} & \longrightarrow & G/G_n \\ & \bar{a} & \longmapsto & \bar{a} + G_n \end{array}$$
The <u>inverse limit of the G/G_n</u> is
$$\varprojlim G/G_n = \{\text{coherent sequences}\}.$$

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the inverse limit of the G/G_n is

$$\varprojlim G/G_n = \{\text{coherent sequences}\}$$

Completions

A Cauchy sequence is a sequence $(a_1, a_2, \ldots)$ such that

if $P \in \mathcal{N}(0)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>\ell}$ then $a_r - a_s \in P$.

Let

$$\hat{G} = \{\text{Cauchy sequences } (a_1, a_2, \ldots)\}$$

with

$$(a_1, a_2, \ldots) = (b_1, b_2, \ldots) \quad \text{if} \quad \lim_{n\to\infty} (b_n - a_n) = 0$$

HW: Show that

$$\begin{array}{rcl} \Psi : \hat{G} & \longrightarrow & \varprojlim G/G_n \\ (a_1, a_2, \ldots) & \longmapsto & (a_1 + G_1, a_2 + G_2, \ldots) \end{array}$$

is an isomorphism.