21.09.2022 (1)
MHS Lect 26

Function spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
Let

$$Map(X,Y) = \{f: X \to Y \mid f \text{ is continuous}\}.$$

The <u>compact open topology</u> on $Map(X,Y)$ is the topology generated by

$$\mathcal{B} = \left\{ B_{K,U} \;\middle|\; \begin{array}{l} K \subseteq X \text{ is compact} \\ U \subseteq Y \text{ is open} \end{array} \right\}$$

where $B_{K,U} = \{f: X \to Y \mid f(K) \subseteq U\}$

A <u>path in X</u> is $p \in Map(\mathbb{R}_{[0,1]}, X)$

A <u>loop in X</u> is $\gamma \in Map(S^1, X)$, where

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

The space $(X, \mathcal{T}_X)$ is <u>path connected</u> if X satisfies:

if $x, y \in X$ then there exists $p \in Map(\mathbb{R}_{[0,1]}, X)$ with $p(0) = x$ and $p(1) = y$

Homotopy

Let $f_1, f_2 \in Map(X,Y)$. The maps f_1 and f_2 are homotopic if there exists $F \in Map(X \times \mathbb{R}_{[0,1]}, Y)$ such that if $x \in X$ then

$$F(x,0) = f_1(x) \quad \text{and} \quad F(x,1) = f_2(x).$$

Write $f_1 \simeq f_2$ if f_1 and f_2 are homotopic.

Define

$$[X,Y] = \frac{Map(X,Y)}{\simeq}$$

Recall: If $(Z, \mathcal{T}_Z)$ is a topolocial space and $\sim$ is an equivalence relation on Z then

$$Z/\sim = \{[z] \mid z \in Z\}$$

(the set of equivalence classes) has the quotient topology, which is the minimal topology such that

$$\begin{array}{ccc} Z & \longrightarrow & Z/\sim \\ z & \longmapsto & [z] \end{array} \quad \text{is continuous.}$$

Here $[z] = \{y \in Z \mid y \sim z\}$.

Based spaces and spheres

A <u>based space</u> (X, x_0) is a topological space $(X, \mathcal{T}_X)$ with a point $x_0 \in X$.

Let (X, x_0) and (Y, y_0) be based spaces.

The space of <u>(based) maps from (X, x_0) to (Y, y_0)</u> is

$$Map((X,x_0),(Y,y_0)) = \left\{ f : X \to Y \;\middle|\; \begin{matrix} f \text{ is continuous} \\ \text{and } f(x_0) = y_0 \end{matrix} \right\}$$

Define

$$[(X,x_0),(Y,y_0)] = \frac{Map((X,x_0),(Y,y_0))}{\simeq}$$

Let $n \in \mathbb{Z}_{\geq 0}$. The <u>n-sphere</u> (S^n, p) is

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n-1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

with the subspace topology coming from $S^n \subseteq \mathbb{R}^{n+1}$ and with the point $p = (0, \ldots, 0)$.

For $n \in \mathbb{Z}_{>0}$ another realization of (S^n, p) is

$$S^n = \frac{\mathbb{R}^n_{[0,1]}}{\left\langle \begin{matrix} (s_1,\ldots,s_n) = (0,\ldots,0) \text{ if there exists} \\ i \in \{1,\ldots,n\} \text{ with } s_i \in \{0,1\} \end{matrix} \right\rangle}$$

with the point $p = (0, \ldots, 0)$.

$S^0 = \{-1, 1\}$, $S^1 =$ , $S^2 =$ 

Homotopy groups and connectedness

Let (X, x_0) be a based space.

The n^{th} homotopy group is

$$\pi_n(X, x_0) = [(S^n, p), (X, x_0)] = \frac{\mathrm{Map}((S^n, p), (X, x_0))}{\simeq}$$

with product

$$(f_1 * f_2)(s_1, \ldots, s_n) = \begin{cases} f_1(2s_1, s_2, \ldots, s_n), & \text{if } 0 \leq s_1 \leq \frac{1}{2} \\ f_2(2s_1 - 1, s_2, \ldots s_n), & \text{if } \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

the identity in $\pi_n(X, x_0)$ is the constant map

$$\begin{aligned} 0 : (S^n, p) &\longrightarrow (X, x_0) \\ (s_1, \ldots, s_n) &\longmapsto x_0 \end{aligned}$$

The fundamental group of (X, x_0) is $\pi_1(X, x_0)$, the group of (homotopy classes) of loops at x_0.

The space (X, x_0) is n-connected if

$$\pi_0(X, x_0) = \{0\}, \pi_1(X, x_0) = \{0\}, \ldots, \pi_n(X, x_0) = \{0\}.$$

The space (X, x_0) is simply connected if (X, x_0) is 1-connected

The space (X, x_0) is path connected if (X, x_0) is 0-connected.