29.07.2022 ①
MHSheet3

Linear transformations

Let V and W be $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

A linear transformation from V to W is a function $T: V \to W$ such that

(a) If $v_1, v_2 \in V$ then $T(v_1+v_2) = T(v_1) + T(v_2)$

(b) If $c \in \mathbb{K}$ and $v \in V$ then $T(cv) = cT(v)$.

Let

$$\mathrm{End}(V, W) = \left\{ T: V \to W \;\middle|\; T \text{ is a linear transformation} \right\}$$

A linear operator is a linear transformation

Try to define a norm on $\mathrm{End}(V, W)$.

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed vector space and $T: V \to W$ a linear transformation. Define

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \;\middle|\; v \in V \right\}$$

$$= \sup \left\{ \|Tv\|_W \;\middle|\; v \in V \text{ and } \|v\|_V = 1 \right\}.$$

Then

$$B(V,W) = \{T : V \to W \mid \|T\| \text{ exists in } \mathbb{R}_{\geq 0}\}$$

is the space of bounded linear operators from V to W.

The addition on $B(V,W)$ is given by

$$(T_1+T_2)(v) = T_1(v)+T_2(v), \quad \text{for } v \in V.$$

The scalar multiplication $\mathbb{K} \times B(V,W) \to B(V,W)$ is given by

$$(cT)(v) = c \cdot T(v), \quad \text{for } v \in V.$$

HW: Show that, if $T_1, T_2 \in B(V,W)$ then

$$\|T_1+T_2\| \leq \|T_1\| + \|T_2\|$$

HW: Show that if $c \in \mathbb{K}$ and $T \in B(V,W)$ then

$$\|cT\| = |c| \cdot \|T\|$$

HW: Show that if $T \in B(V,W)$ and $\|T\| = 0$ then $T = 0$.

Theorem If V is complete then $B(V,W)$ is complete.

Dual spaces

Let V be a $\mathbb{K}$-vector space, where $\mathbb{K}$ is a field.

The dual space is

$$V^* = \{\varphi : V \to \mathbb{K} \mid \varphi \text{ is a linear operator}\}$$

with addition $V^* \times V^* \to V^*$ given by

$$(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v), \quad \text{for } v \in V,$$

and scalar multiplication $\mathbb{K} \times V^* \to V^*$ given by

$$(c\varphi)(v) = c \cdot \varphi(v), \quad \text{for } v \in V.$$

Let $(V, \| \|)$ be a normed $\mathbb{K}$-vector space, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

The dual space is

$$V^* = \mathcal{B}(V, \mathbb{K}) = \left\{\varphi : V \to \mathbb{K} \;\middle|\; \begin{array}{l}\varphi \text{ is a linear transf.} \\ \text{and } \|\varphi\| \text{ exists in } \mathbb{R}_{\geq 0}\end{array}\right\}$$

Elements of $\mathcal{B}(V, \mathbb{K})$ are sometimes called bounded linear functionals.

The favorite result (in every book) is:

<u>Proposition</u> Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed $\mathbb{K}$-vector spaces. Let $T: V \to W$ be a linear operator.

The following are equivalent:

(a) T is bounded

(b) T is continuous

(c) T is uniformly continuous.

<u>Definitions</u> Let $\mathbb{E} = \{10^{-1}, 10^{-2}, 10^{-3}, \dots\}$.

- $T: V \to W$ is <u>continuous</u> if T satisfies:

  if $x \in V$ and $\varepsilon \in \mathbb{E}$ then
  there exists $\delta \in \mathbb{E}$ such that
  if $y \in V$ and $d_V(x, y) < \delta$ then $d_W(f(x), f(y)) < \varepsilon$

- $T: V \to W$ is <u>uniformly continuous</u> if T satisfies:

  if $\varepsilon \in \mathbb{E}$ then
  there exists $\delta \in \mathbb{E}$ such that
  if $x \in V$ and $y \in V$ and $d_V(x, y) < \delta$
  then $d_W(f(x), f(y)) < \varepsilon$.