17.10.2022 (1)
MHS Lect 34

Subspaces of $\mathbb{R}^\infty$

$$\mathbb{R}^\infty = \{\text{sequences } (x_1, x_2, \ldots) \text{ in } \mathbb{R}\}$$
$$= \{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R}\}$$
$$= \{x : \mathbb{Z}_{>0} \to \mathbb{R}\}.$$

Let

$$c_c = \{x \in \mathbb{R}^\infty \mid \text{all but a finite number of the } x_i \text{ are } 0\}$$

$$\ell^1 = \{x \in \mathbb{R}^\infty \mid \|x\|_1 \text{ exists in } \mathbb{R}\}$$

$$\ell^p = \{x \in \mathbb{R}^\infty \mid \|x\|_p \text{ exists in } \mathbb{R}_{\geq 0}\}$$

$$c_0 = \{x \in \mathbb{R}^\infty \mid \lim_{n \to \infty} x_n = 0\}$$

$$\ell^\infty = \{x \in \mathbb{R}^\infty \mid \|x\|_\infty \text{ exists in } \mathbb{R}_{\geq 0}\}.$$

where

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}, \quad \text{for } p \in \mathbb{R}_{\geq 1},$$

$$\|x\|_\infty = \sup\{|x_i| \mid i \in \mathbb{Z}_{>0}\}$$

<u>Claim:</u> Let $p, q \in \mathbb{R}_{>1}$ with $p < 2 < q$ and $\frac{1}{p} + \frac{1}{q} = 1$

Then

$$\ell^1 \subsetneq \ell^p \subsetneq \ell^2 \subsetneq \ell^q \subsetneq c_0 \subsetneq \ell^\infty$$

<u>Inclusions</u> 17.10.2022 2

Assume $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$ is such that MHS Lect3

there exists $N \in \mathbb{Z}_{>0}$ such that

if $i \in \mathbb{Z}_{>N}$ then $|x_i| < 1$.

If $i \in \mathbb{Z}_{>N}$ then

$$|x_i| > |x_i|^p > |x_i|^2 > |x_i|^q \quad \text{giving}$$

$$\sum_{n=N+1}^{\infty} |x_i| \geq \sum_{n=N+1}^{\infty} |x_i|^p \geq \sum_{n=N+1}^{\infty} |x_i|^2 \geq \sum_{n=N+1}^{\infty} |x_i|^q.$$

Thus

$$\ell^1 \subseteq \ell^p \subseteq \ell^2 \subseteq \ell^q \subseteq c_0$$

where the last inclusion follows from

if $\sum_{n=N+1}^{\infty} |x_i|^q$ exists in $\mathbb{R}_{\geq 0}$ then $\lim_{n\to\infty} |x_n| = 0$.

If $\lim_{n\to\infty} |x_n| = 0$ then $\sup\{|x_n| \mid n \in \mathbb{Z}_{>0}\}$ exists in $\mathbb{R}_{\geq 0}$ and so

$$c_0 \subseteq \ell^\infty.$$

Thus $\ell^1 \subseteq \ell^p \subseteq \ell^2 \subseteq \ell^q \subseteq c_0 \subseteq \ell^\infty$

Strict inclusions

Let $p,q\in\mathbb{R}_{>1}$ with $p<q$. Then $0<\frac{1}{q}<\frac{p}{q}<1<2$ and, using that

$$\sum_{n=1}^{\infty}\frac{1}{n^s}\ \text{converges in } \mathbb{R} \text{ for } s\in\mathbb{R}_{>1} \text{ and}$$
$$\text{does not converge in } \mathbb{R} \text{ for } s\in\mathbb{R}_{(0,1]},$$

gives that:

If $x=\left(\frac{1}{2^2},\frac{1}{3^2},\dots\right)$ then $x\in\ell^1$ and $x\notin c_c$;

If $x=\left(\frac{1}{2},\frac{1}{3},\dots\right)$ then $x\in\ell^p$ and $x\notin\ell^1$;

If $x=\left(\frac{1}{2^{p/q}},\frac{1}{3^{p/q}},\dots\right)$ then $x\in\ell^q$ and $x\notin\ell^p$;

If $x=\left(\frac{1}{2^{1/q}},\frac{1}{3^{1/q}},\dots\right)$ then $x\in c_0$ and $x\notin\ell^q$;

If $x=(1,1,\dots)$ then $x\in\ell^\infty$ and $x\notin c_0$.

So
$$c_c\subsetneq\ell^1\subsetneq\ell^p\subsetneq\ell^2\subsetneq\ell^q\subsetneq c_0\subsetneq\ell^\infty,$$
for $p,q\in\mathbb{R}_{>1}$ with $p<2<q$ and $\frac{1}{p}+\frac{1}{q}=1$.

Closures

Let $p \in \mathbb{R}_{\geq 1}$. Let $e_i = (0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the i^{th} entry. Then

$$C_c = \mathrm{span}\{e_1, e_2, \ldots\}.$$

Since finite sums always exist in $\mathbb{R}$ then

$$C_c \subseteq \ell^p.$$

Let $x = (x_1, x_2, \ldots) \in \ell^p$. ~~Let~~ For $k \in \mathbb{Z}_{>0}$ let

$$x_{\leq k} = (x_1, x_2, \ldots, x_k, 0, 0, \ldots) = \sum_{i=1}^{k} x_i e_i \quad \in C_c.$$

Since $x \in \ell^p$ then

$$\lim_{k \to \infty} \|x - x_{\leq k}\|_p^p = \lim_{k \to \infty} \left(\sum_{n=1}^{\infty} |x_n - (x_{\leq k})_n|^p \right)$$

$$= \lim_{k \to \infty} \left(\sum_{n=k}^{\infty} |x_n|^p \right) = 0.$$

So $x = \lim_{k \to \infty} x_{\leq k}$, with respect to $\| \ \|_p$.

So $x \in \overline{C_c}$ and $\ell^p \subseteq \overline{C_c}$. Thus,

with respect to $\| \ \|_p$, $\quad \overline{C_c} = \ell^p$

for $p \in \mathbb{R}_{\geq 1}$.

If $x = (x_1, x_2, \ldots) \in c_0$ then $\lim_{n \to \infty} x_n = 0$. MHS Lect 34

So, if $k \in \mathbb{Z}_{>0}$ then there exists $N_k \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>N_k}$ then $|x_n| < 10^{-k}$.

Hence, if $n \in \mathbb{Z}_{>N_k}$ then

$$\|x - x_{\leq n}\|_\infty = \sup\{|x_i| \mid i \in \mathbb{Z}_{>n}\} < 10^{-k}.$$

So $\lim_{k \to \infty} \|x - x_{\leq k}\|_\infty = 0$ and thus, with respect to $\| \|_\infty$, $\lim_{k \to \infty} x_{\leq k} = x$.

Thus, with respect to $\| \|_\infty$

$$\overline{c_c} = c_0.$$