Measurable spaces and functions

A <u>measurable space</u> is a set X with a collection $\mathcal{M}$ of subsets of X such that

(a) $X \in \mathcal{M}$

(b) If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$.

(c) If $A_1, A_2, \ldots \in \mathcal{M}$ then $A_1 \cup A_2 \cup \cdots \in \mathcal{M}$.

Let $(X, \mathcal{M})$ be a measurable space.

A <u>simple</u> <u>measurable function</u> is an element of $\mathrm{span}\{\mathbb{1}_A \mid A \in \mathcal{M}\}$, where

$$\mathbb{1}_A : X \to \mathbb{R}_{\geq 0} \text{ is given by } \mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let $(Y, \mathcal{T}_Y)$ be a topological space.

A <u>measurable function from X to Y</u> is a function $f : X \to Y$ such that

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{M}$.

Integration

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Let $(X, \mathcal{M})$ be a measure space. MHS Lect 35

A positive measure on X is a function

$$\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\} \text{ such that}$$

if $A_1, A_2, \ldots \in \mathcal{M}$ satisfy

if $i, j \in \mathbb{Z}_{>0}$ and $i \neq j$ then $A_i \cap A_j = \phi$

then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Let μ be a positive measure on X.

For a simple measurable function

$$s = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i} \quad \text{and } E \in \mathcal{M}$$

define

$$\int_E s \, d\mu = \sum_{i=1}^{n} c_i \, \mu(A_i \cap E).$$

For a measurable function $f : X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ define

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple} \\ \text{measurable and} \\ 0 \leq s \leq f \end{array} \right\}$$

Integration of real and complex functions

For a function $f : X \to \mathbb{R}_{[-\infty,\infty]}$ define

$$f^+ = \frac{1}{2}(|f| + f) \quad \text{and} \quad f^- = \frac{1}{2}(|f| - f)$$

so that

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ 0, & \text{if } f(x) < 0, \end{cases} \quad \text{and}$$

$$f^-(x) = \begin{cases} 0, & \text{if } f(x) > 0, \\ -f(x), & \text{if } f(x) \leq 0. \end{cases}$$

For a measurable function $f : X \to \mathbb{R}_{[-\infty,\infty]}$ such that

$$\int_E f^+ \, d\mu < \infty \quad \text{or} \quad \int_E f^- \, d\mu < \infty$$

define

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu .$$

For a measurable function $f : X \to \mathbb{C}$ let

$f = u + iv$, where $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ are measurable and define

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu$$

Function spaces $\mathcal{L}^p(\mu)$, $\mathcal{L}^\infty(\mu)$, $C_c(X)$ and $C_0(X)$.

Let $(X, \mathcal{M})$ be a measurable space and

$\mu : \mathcal{M} \to \mathbb{R}_{[0,\infty]}$ a positive measure on $\mathcal{M}$.

Define

$$C_c(X) = \left\{ f : X \to \mathbb{C} \ \middle|\ \begin{array}{l} f \text{ is continuous and} \\ \text{supp}(f) \text{ is compact} \end{array} \right\}$$

$$\mathcal{L}^p(\mu) = \left\{ f : X \to \mathbb{C} \ \middle|\ \begin{array}{l} f \text{ is measurable and } \|f\|_p \\ \text{exists in } \mathbb{R}_{\geq 0} \end{array} \right\}$$

$$C_0(X) = \left\{ f : X \to \mathbb{C} \ \middle|\ \begin{array}{l} f \text{ is continuous and} \\ f \text{ vanishes at infinity} \end{array} \right\}$$

$$\mathcal{L}^\infty(\mu) = \left\{ f : X \to \mathbb{C} \ \middle|\ \begin{array}{l} f \text{ is measurable and} \\ \|f\|_\infty \text{ exists in } \mathbb{R}_{\geq 0} \end{array} \right\}$$

where

$$\text{supp}(f) = \overline{\{ x \in X \mid f(x) \neq 0 \}}$$

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

$$\|f\|_\infty = \inf \{ \alpha \in \mathbb{R} \mid \mu(f^{-1}(\mathbb{R}_{[\alpha,\infty]})) = 0 \}$$

and <u>f vanishes at infinity</u> if f satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists a compact set $K \subseteq X$ such that if $x \notin K$ then $|f(x)| < \varepsilon$.