2.1.4 The normed vector space $(\ell^q)^*$

Use the notations $e_i = (0, \ldots, 0, 1, 0, 0, \ldots)$ where the 1 is in the ith spot, and

$$\varphi_x(y) = \langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i, \qquad \text{for} \quad x = (x_1, x_2, \ldots) \text{ and } y = (y_1, y_2, \ldots).$$

Let $p, q \in \mathbb{R}_{>1}$ with $\frac{1}{p} + \frac{1}{q} = 1$. The Hölder-Minkowski inequality (REFERENCE) says that

$$\text{if } x \in \ell^p \text{ and } y \in \ell^q \text{ then} \qquad |\langle x, y\rangle| \le \|x\|_p \|y\|_q.$$

If $\varphi \in (\ell^q)^*$ and $y \in \ell^q$ then

$$|\varphi(y)| = \Big|\varphi\Big(\sum_{i=1}^{\infty} y_i e_i\Big)\Big| = \Big|\sum_{i=1}^{\infty} y_i \varphi(e_i)\Big| = |\langle y, (\varphi(e_1), \varphi(e_2), \ldots)\rangle| \le \|y\|_q \cdot \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p,$$

which gives

$$\|\varphi\| = \sup\{|\varphi(y)| \mid y \in \ell^q \text{ and } \|y\|_q = 1\} \le \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p.$$

Let

$$s = (s_1, s_2, \ldots), \qquad \text{where} \quad s_i = \mathrm{sgn}(\varphi(e_i)) \cdot |\varphi(e_i)|^{\frac{p}{q}}.$$

Then

$$\begin{aligned}|\varphi(s)| &= \Big|\varphi\Big(\sum_{i=1}^{\infty} s_i e_i\Big)\Big| = \Big|\sum_{i=1}^{\infty} s_i \varphi(e_i)\Big| = \Big|\sum_{i=1}^{\infty} |\varphi(e_i)||\varphi(e_i)|^{\frac{p}{q}}\Big| = \Big|\sum_{i=1}^{\infty} |\varphi(e_i)||\varphi(e_i)|^{p(1-\frac{1}{p})}\Big| \\ &= \Big(\sum_{i=1}^{\infty} |\varphi(e_i)|^p\Big)^{\frac{1}{p}+\frac{1}{q}} = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p \Big(\sum_{i=1}^{\infty} \big(|\varphi(e_i)|^{\frac{p}{q}}\big)^q\Big)^{\frac{1}{q}} = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p \cdot \|s\|_q,\end{aligned}$$

which gives that $\|\varphi\| \ge \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p$. Hence, if $\varphi \in (\ell^q)^*$ then

$$\|\varphi\| = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p.$$

If $x \in \ell^p$ then

$$\|\varphi_x\| = \|x\|_p, \qquad \text{because } x = (x_1, x_2, \ldots) = (\varphi_x(e_1), \varphi_x(e_2), \ldots).$$

These computations show that there are well defined isometries (of normed vector spaces)

$$\begin{array}{rccc} \Phi_p\colon & \ell^p & \longrightarrow & (\ell^q)^* \\ & x & \longmapsto & \varphi_x \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \Psi_p\colon & (\ell^q)^* & \to & \ell^p \\ & \varphi & \mapsto & (\varphi(e_1), \varphi(e_2), \ldots). \end{array}$$

If $\varphi \in (\ell^q)^*$ then φ is determined by the values $x_i = \varphi(e_i)$ because φ is continuous and $\overline{c_c} = \ell^q$. Thus $\varphi = \varphi_x = \Phi_p(x)$ and Φ_p is surjective. Since $(\varphi_x(e_1), \varphi_x(e_2), \ldots) = (x_1, x_2, \ldots) = x$ the functions Ψ_p and Φ_p are inverse functions. Since Φ_p and Ψ_p are isometries then $\|\Phi_p\| = 1$ and $\|\Psi_p\| = 1$. In summary, we might abuse notation and write

$$(\ell^q)^* = \ell^p.$$

2.1.5 The normed vector space $(c_0)^*$

Let $x \in \ell^1$ and $y \in \ell^\infty$. Then

$$\begin{aligned}|\langle x, y\rangle| &= \Big\| \sum_{i=1}^{\infty} x_i y_i \Big\| = \Big\| \lim_{k\to\infty} \sum_{i=1}^{k} x_i y_i \Big\| = \lim_{k\to\infty} \Big\| \sum_{i=1}^{k} x_i y_i \Big\| \le \lim_{k\to\infty} \sum_{i=1}^{k} |x_i y_i| \\ &\le \lim_{k\to\infty} \|y\|_\infty \sum_{i=1}^{k} |x_i| = \|y\|_\infty \cdot \|x\|_1,\end{aligned}$$

which is an analogue of the Hölder-Minkoswki inequality. If $\varphi \in (c_0)^*$ and $y \in c_0$ then

$$|\varphi(y)| = \Big\| \varphi\Big(\sum_{i=1}^{\infty} y_i e_i\Big) \Big| = \Big| \sum_{i=1}^{\infty} y_i \varphi(e_i) \Big| = |\langle y, (\varphi(e_1), \varphi(e_2), \ldots)\rangle| \le \|y\|_\infty \cdot \|(\varphi(e_1), \varphi(e_2), \ldots)\|_1,$$

which gives

$$\|\varphi\| = \sup\{|\varphi(y)| \mid y \in c_0 \text{ and } \|y\|_\infty = 1\} \le \|(\varphi(e_1), \varphi(e_2), \ldots)\|_1.$$

Let $s = (s_1, s_2, \ldots)$ where $s_i = \mathrm{sgn}(\varphi(e_i))$. Then

$$\begin{aligned}|\varphi(s)| &= \Big| \varphi\Big(\sum_{i=1}^{\infty} s_i e_i\Big) \Big| = \Big| \sum_{i=1}^{\infty} s_i \varphi(e_i) \Big| = \sum_{i=1}^{\infty} |\varphi(e_i)|| \\ &= \|(\varphi(e_1), \varphi(e_2), \ldots)\|_1 = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_1 \cdot \|s\|_\infty,\end{aligned}$$

which gives that $\|\varphi\| \ge \|(\varphi(e_1), \varphi(e_2), \ldots)\|_p$. Hence, if $\varphi \in (c_0)^*$ then

$$\|\varphi\| = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_1.$$

If $x \in \ell^1$ then

$$\|\varphi_x\| = \|x\|_1, \qquad \text{because } x = (x_1, x_2, \ldots) = (\varphi_x(e_1), \varphi_x(e_2), \ldots).$$

These computations show that there are well defined isometries (of normed vector spaces)

$$\begin{array}{cccc} \Phi_1\colon & \ell^1 & \longrightarrow & (c_0)^* \\ & x & \longmapsto & \varphi_x \end{array} \quad \text{and} \quad \begin{array}{cccc} \Psi_1\colon & (c_0)^* & \to & \ell^1 \\ & \varphi & \mapsto & (\varphi(e_1), \varphi(e_2), \ldots). \end{array}$$

If $\varphi \in (c_0)^*$ then φ is determined by the values $x_i = \varphi(e_i)$ because φ is continuous and $\overline{c_c} = c_0$. Thus $\varphi = \varphi_x = \Phi_1(x)$ and Φ_1 is surjective. Since $(\varphi_x(e_1), \varphi_x(e_2), \ldots) = (x_1, x_2, \ldots) = x$ the functions Ψ_1 and Φ_1 are inverse functions. Since Φ_1 and Ψ_1 are isometries then $\|\Phi_1\| = 1$ and $\|\Psi_1\| = 1$. In summary, we might abuse notation and write

$$(c_0)^* = \ell^1.$$

2.1.6 The normed vector space $(\ell_1)^*$

If $\varphi \in (\ell^1)^*$ and $y \in \ell^\infty$ then

$$|\varphi(y)| = \Big\|\varphi\Big(\sum_{i=1}^\infty y_i e_i\Big)\Big| = \Big|\sum_{i=1}^\infty y_i\varphi(e_i)\Big| = |\langle y, (\varphi(e_1), \varphi(e_2), \ldots)\rangle| \le \|y\|_1 \cdot \|(\varphi(e_1), \varphi(e_2), \ldots)\|_\infty,$$

which gives

$$\|\varphi\| = \sup\{|\varphi(y)| \mid y \in \ell^1 \text{ and } \|y\|_1 = 1\} \le \|(\varphi(e_1), \varphi(e_2), \ldots)\|_\infty.$$

If $i \in \mathbb{Z}_{>0}$ then $\|\varphi\| \ge |\varphi(e_i)|$, and so $\|\varphi\| \ge \|(\varphi(e_1), \varphi(e_2), \ldots)\|_\infty$. Hence, if $\varphi \in (\ell^1)^*$ then

$$\|\varphi\| = \|(\varphi(e_1), \varphi(e_2), \ldots)\|_\infty.$$

If $x \in \ell^1$ then

$$\|\varphi_x\| = \|x\|_\infty, \qquad \text{because } x = (x_1, x_2, \ldots) = (\varphi_x(e_1), \varphi_x(e_2), \ldots).$$

These computations show that there are well defined isometries (of normed vector spaces)

$$\begin{array}{rccc} \Phi_\infty\colon & \ell^\infty & \longrightarrow & (\ell^1)^* \\ & x & \longmapsto & \varphi_x \end{array} \quad \text{and} \quad \begin{array}{rccc} \Psi_\infty\colon & (\ell^1)^* & \to & \ell^\infty \\ & \varphi & \mapsto & (\varphi(e_1), \varphi(e_2), \ldots). \end{array}$$

If $\varphi \in (\ell^1)^*$ then φ is determined by the values $x_i = \varphi(e_i)$ because φ is continuous and $\overline{c_c} = \ell^1$. Thus $\varphi = \varphi_x = \Phi_\infty(x)$ and Φ_∞ is surjective. Since $(\varphi_x(e_1), \varphi_x(e_2), \ldots) = (x_1, x_2, \ldots) = x$ the functions Ψ_∞ and Φ_∞ are inverse functions. Since Φ_∞ and Ψ_∞ are isometries then $\|\Phi_\infty\| = 1$ and $\|\Psi_\infty\| = 1$. In summary, we might abuse notation and write

$$(\ell^1)^* = \ell^\infty.$$

2.1.7 Completeness

Let $p, q \in \mathbb{R}_{>1}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Since

$$\ell^1 = (c_0)^* = B(c_0, \mathbb{R}), \qquad \ell^p = (\ell^q)^* = B(\ell^q, \mathbb{R}), \qquad \ell^\infty = (\ell^1)^* = B(\ell^1, \mathbb{R}),$$

then the theorem that tells us that if W is complete then $B(V, W)$ is complete tells us that

$$\ell^1, \ell^p \text{ and } \ell^\infty \quad \text{are all complete.}$$

Since ℓ^1, ℓ^p and ℓ^∞ are all complete and Cauchy sequences converge in a complete metric space then $\widehat{c_c} = \overline{c_c}$ in ℓ^1, ℓ^p and ℓ^∞. Thus

$$\widehat{c_c} = \overline{c_c} = \ell^1 \quad \text{in } \ell^1, \qquad \widehat{c_c} = \overline{c_c} = \ell^p \quad \text{in } \ell^p, \quad \text{and} \quad \widehat{c_c} = \overline{c_c} = c_0 \quad \text{in } \ell^\infty.$$