Uses of $\mathbb{R}_{\geq 0}$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A normed $\mathbb{K}$ vector space is a $\mathbb{K}$-vector space V with a function $\| \ \|: V \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$,

(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c| \cdot \|v\|$,

(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

Define $d: V \times V \to \mathbb{R}_{\geq 0}$ by

$$d(x,y) = \|y - x\|.$$

A metric space is a set X with a function $d: X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$,

(b) If $x, y \in X$ then $d(x,y) = d(y,x)$

(c) If $x \in X$ then $d(x,x) = 0$.

(d) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$.

A normed vector space is a very special kind of metric space.

What is $\mathbb{R}_{\geq 0}$?

$\mathbb{R}_{\geq 0}$ is the set of decimal expansions.

$$\mathbb{R}_{\geq 0} = \{ z.d_1 d_2 \ldots \mid z \in \mathbb{Z}_{\geq 0},\ d_i \in \{0, \ldots, 9\} \}$$

with

$$z.999\ldots = (z+1).000\ldots \quad \text{if } z \in \mathbb{Z}_{\geq 0} \text{ and}$$

$$z.d_1 d_2 \cdots d_{k-1} d_k 999\ldots = z.d_1 \cdots d_{k-1}(d_k+1)000\ldots$$

if $z \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{>0}$ and $d_k \neq 9$.

<u>Question</u> How do you say how to add and multiply in $\mathbb{R}_{\geq 0}$?

The space $\mathbb{Q}_{\geq 0}$

$$\mathbb{Q}_{\geq 0} = \left\{ \frac{r}{s} \;\middle|\; r, s \in \mathbb{Z}_{\geq 0} \text{ and } s \neq 0 \right\}.$$

with

$$\frac{r}{s} = \frac{p}{q} \quad \text{if} \quad rq = ps.$$

Define addition $\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + s_1 r_2}{s_1 s_2}.$$

Define multiplication $\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ by

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

Define an order on $\mathbb{Q}_{\geq 0}$ by

$x \leq y$ if there exists $z \in \mathbb{Q}_{\geq 0}$ such that

$$x + z = y.$$

Define $d: \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ by

$d(x,y) = z$ if $z \in \mathbb{Q}_{\geq 0}$ satisfies

$$x + z = y \text{ or } y + z = x.$$

The completion $\hat{\mathbb{Q}}_{\geq 0}$

is the space of Cauchy sequences in $\mathbb{Q}_{\geq 0}$,

$$\hat{\mathbb{Q}}_{\geq 0} = \{\text{Cauchy sequences } (a_1, a_2, \ldots) \text{ in } \mathbb{Q}_{\geq 0}\}$$

with

$$(a_1, a_2, \ldots) = (b_1, b_2, \ldots) \text{ if } \lim_{n \to \infty} d(a_n, b_n) = 0.$$

Define addition on $\hat{\mathbb{Q}}_{\geq 0}$ by

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$$

Define multiplication on $\hat{\mathbb{Q}}_{\geq 0}$ by

$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots)$$

Define $\iota: \mathbb{Q}_{\geq 0} \to \hat{\mathbb{Q}}_{\geq 0}$ by

$$\iota(a) = (a, a, a, \ldots).$$

Define an <u>order</u> on $\hat{\mathbb{Q}}_{\geq 0}$ by

$x \leq y$ if there exists $z \in \hat{\mathbb{Q}}_{\geq 0}$ such that $x + z = y$.

Define $\hat{d} : \hat{\mathbb{Q}}_{\geq 0} \times \hat{\mathbb{Q}}_{\geq 0} \longrightarrow \hat{\mathbb{Q}}_{\geq 0}$ by

$$\hat{d}((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = (d(a_1, b_1), d(a_2, b_2), \ldots)$$

<u>Do you believe $\mathbb{R}_{\geq 0} = \hat{\mathbb{Q}}_{\geq 0}$?</u>

Is there a bijection $\Phi : \mathbb{R}_{\geq 0} \longrightarrow \hat{\mathbb{Q}}_{\geq 0}$?

Think about what a decimal expansion really is.

$$z.d_1 d_2 d_3 \cdots = z + d_1 10^{-1} + d_2 10^{-2} + d_3 \cdot 10^{-3} + \cdots$$

$$= z + \sum_{k=1}^{\infty} d_k 10^{-k} = (z, z + s_1, z + s_2, \ldots)$$

where $s_\ell = \sum_{k=1}^{\ell} d_k 10^{-k} = d_1 \cdot 10^{-1} + d_2 10^{-2} + \cdots + d_\ell 10^{-\ell}$

So a decimal expansion <u>is</u> a Cauchy sequence in $\mathbb{Q}_{\geq 0}$ and

$$\Phi : \mathbb{R}_{\geq 0} \longrightarrow \hat{\mathbb{Q}}_{\geq 0}$$

$$z.d_1 d_2 \ldots \longmapsto (z, z + s_1, z + s_2, \ldots).$$

Is Φ a bijection?

What is the inverse map $\Phi^{-1} : \hat{\mathbb{Q}}_{\geq 0} \to \mathbb{R}_{\geq 0}$?