Hilbert spaces

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, 10^{-3}, \ldots\}$ and let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

Let V be a $\mathbb{K}$-vector space.

A positive definite Hermitian form on V is a function $\langle , \rangle : V \times V \to \mathbb{K}$ such that

(a) If $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in V$ then

$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ and

$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$.

(b) If $c \in \mathbb{K}$ and $x, y \in V$ then

$\langle cx, y \rangle = c\langle x, y \rangle$ and $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$.

(c) If $x \in V$ then $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$

(d) If $x \in V$ and $\langle x, x \rangle = 0$ then $x = 0$.

Define $\| \; \| : V \to \mathbb{R}_{\geq 0}$ by

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \text{for } x \in V$$

Define $d : V \times V \to \mathbb{R}_{\geq 0}$ by

$$d(x, y) = \|y - x\|$$

For $\varepsilon \in \mathbb{E}$ and $x \in V$ define

$$B_\varepsilon(x) = \{y \in V \mid d(y,x) < \varepsilon\}$$

For $\varepsilon \in \mathbb{E}$ define

$$\underline{B}_\varepsilon = \{(x,y) \in V \times V \mid d(y,x) < \varepsilon\}.$$

A <u>Hilbert space</u> is a $\mathbb{K}$-vector space V with a positive definite Hermitian form $\langle , \rangle : V \times V \to \mathbb{K}$ which is complete (as a normed vector space).

A <u>metric space</u> is a set V with a function $d : V \times V \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y, z \in V$ then $d(x,y) \leq d(x,z) + d(z,y)$

(b) If $x, y \in V$ then $d(x,y) = d(y,x)$

(c) If $x \in V$ then $d(x,x) = 0$

(d) If $x, y \in V$ ~~then~~ and $d(x,y) = 0$ then $x = y$.

Let (V, d_V) and (W, d_W) be metric spaces.
An <u>isometry from V to W</u> is a function $\varphi : V \to W$ such that

if $x, y \in V$ then $d_V(x,y) = d_W(\varphi(x), \varphi(y))$.

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed vector spaces. An <u>isometry from V to W</u> is a function $\Phi: V \to W$ such that

if $x \in V$ then $\|x\|_V = \|\Phi(x)\|_W$.

Let $(V, \langle , \rangle_V)$ and $(W, \langle , \rangle_W)$ be Hilbert spaces. An <u>isometry from V to W</u> is a function $\Phi: V \to W$ such that

$$\langle x, y\rangle_V = \langle \Phi(x), \Phi(y)\rangle_W$$

<u>Dual spaces</u>

Let $(V, \|\ \|)$ be a normed $\mathbb{K}$-vector space. The <u>dual of V</u> is the $\mathbb{K}$-vector space

$$V^* = \left\{ \varphi: V \to \mathbb{K} \;\middle|\; \begin{array}{l} \varphi \text{ is a linear transformation} \\ \text{and } \|\varphi\| \text{ exists in } \mathbb{R}_{\geq 0} \end{array} \right\}$$

where

$$\|\varphi\| = \sup\left\{ \frac{|\varphi(v)|}{\|v\|} \;\middle|\; v \in V \text{ and } v \neq 0 \right\}$$

There is no "obvious" inner product on V^*

Proposition Let $(V, \langle\ \rangle)$ be a $\mathbb{K}$-vector space with a positive definite Hermitian form. Assume $k \in \mathbb{Z}_{>0}$ and $\dim(V) = k$. Then

$$\begin{aligned} \Phi : V &\longrightarrow V^* \\ x &\longmapsto \varphi_x : V \to \mathbb{K} \\ &\qquad\quad v \mapsto \langle v, x\rangle \end{aligned}$$

is a bijective skew-linear transformation.

Let V and W be $\mathbb{K}$-vector spaces.

A skew linear transformation from V to W is a function $\Phi : V \longrightarrow W$ such that

(a) If $x_1, x_2 \in V$ then $\Phi(x_1 + x_2) = \Phi(x_1) + \Phi(x_2)$

(b) If $c \in \mathbb{K}$ and $x \in V$ then $\Phi(cx) = \bar{c}\,\Phi(x)$.

Theorem Let $(H, \langle\ \rangle)$ be a Hilbert space. Then

$$\begin{aligned} \Phi : H &\longrightarrow H^* \\ x &\longmapsto \varphi_x : H \to \mathbb{K} \\ &\qquad\quad h \mapsto \langle h, x\rangle \end{aligned}$$

is a bijective skew-linear isometry and

$$\|\Phi\| = 1$$