23 Problem list: Function spaces and number systems

23.1 Properties of $\mathbb{R}_{\geq 0}$

1. ($\mathbb{R}_{\geq 0}$ is an ordered commutative monoid with mutliplication)

   (a) Show that if $x, y, z \in \mathbb{R}_{\geq 0}$ and $x \leq y$ then $x + z \leq y + z$.
   (b) Show that if $x, y \in \mathbb{R}_{\geq 0}$ then $xy \in \mathbb{R}_{\geq 0}$

2. (Continuity of the operations in $\mathbb{R}_{\geq 0}$) Let $X = \mathbb{R}_{\geq 0}$ with metric given by $d(x, y) = |x - y|$. Show that the functions

$$\begin{matrix} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & x + y \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & -x \end{matrix} \quad \text{are uniformly continuous}$$

   and the function

$$\begin{matrix} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & xy \end{matrix} \qquad \text{is continuous but not uniformly continuous.}$$

3. (numbers gone missing)

   (a) Show that $\sqrt{2} \notin \mathbb{Q}_{\geq 0}$.
   (b) Show that $\pi \notin \mathbb{Q}_{\geq 0}$.
   (c) Show that $e \notin \mathbb{Q}_{\geq 0}$.
   (d) Show that $\sqrt{-1} \notin \mathbb{Q}$.
   (e) Show that $\sqrt{-1} \notin \mathbb{R}$.

4. (The topology on $\mathbb{Q}_{\geq 0}$) Show that $\mathbb{Q}_{\geq 0}$ is not discrete and that $\mathbb{Q}_{\geq 0}$ is Hausdorff. Show that $\mathbb{Q}_{\geq 0}$ is not a complete uniform space.

5. (The uniformity on $\mathbb{Q}$)

   (a) Show that $\mathbb{Q}_{\geq 0}$ is not a complete uniform space.
   (b) Show that $|x|$, $x^+ = \sup(x, 0)$ and $x^- = \sup(-x, 0)$ are uniformly continuous on $\mathbb{Q}$.
   (c) Show that $\sup(x, y)$ and $\inf(x, y)$ are uniformly continuous on $\mathbb{Q} \times \mathbb{Q}$.

6. ($\mathbb{R}_{\geq 0}$ is complete) Show that $\mathbb{R}_{\geq 0}$ is a complete metric space.

7. (Intervals in $\mathbb{R}_{\geq 0}$) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$.

   (a) Show that $[a, b]$ is closed in $\mathbb{R}_{\geq 0}$.
   (b) Show that (a, b) is open in $\mathbb{R}_{\geq 0}$.

8. (Using intervals to determine the topology on $\mathbb{R}$)

(a) Show that if $U \subseteq \mathbb{R}$ and $0 \in U$ and U is open then U contains a subset in $\{[-r, r] \mid r \in \mathbb{Q}_{>0}\}$.
(b) Show that if $x \in R$ and $U \subseteq \mathbb{R}$ and $x \in U$ and U is open then U contains a subset in $\{[x - r, x + r] \mid r \in \mathbb{Q}_{>0}\}$.

9. (Rationals between reals) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$.

   (a) Show that there exists a rational number $r \in \mathbb{Q}_{\geq 0}$ with $a < r < b$.
   (b) Show that there exists an irrational number $r \in \mathbb{Q}^c_{\geq 0}$ with $a < r < b$.

10. (The Archimedean axiom holds) If $x, y \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.

11. (increasing bounded sequences converge) Let $(a_1, a_2, \ldots)$ be a sequence in $\mathbb{R}$ such that $a_1 \leq a_2 \leq \cdots$ and there exists $b \in \mathbb{R}$ such that if $i \in \mathbb{Z}_{>0}$ then $a_i < b$. Show that $\lim_{n\to\infty} a_n$ exists, $\sup\{a_1, a_2, \ldots\}$ exists and
$$\lim_{n\to\infty} a_n = \sup\{a_1, a_2, \ldots\}.$$

12. (The Heine-Borel or Borel-Lebesgue theorem) Let $A \subseteq \mathbb{R}$. Show that A is compact if and only if A is closed and bounded.

13. ($\mathbb{R}$ is locally compact) Show that $\mathbb{R}$ is locally compact and not compact.

14. ($\mathbb{R}_{\geq 0}$ is locally compact) Show that $\mathbb{R}_{\geq 0}$ is locally compact and that the one-point compactification $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is compact. Conclude that every sequence in $\mathbb{R}_{\geq 0}$ has a cluster point in $\mathbb{R}_{\geq 0} \cup \{\infty\}$.

15. (The least upper bound property) Let $A \subseteq \mathbb{R}$ with $A \neq \emptyset$ and A bounded above. Then $\sup(A)$ exists.

16. (connected subsets of $\mathbb{R}$ are intervals) Let $A \subseteq \mathbb{R}$. Show that A is connected if and only if A is an interval.

17. (continuous injective functions are strictly monotonic) Let I be an interval in $\mathbb{R}_{\geq 0}$ and let $f : I \to \mathbb{R}_{\geq 0}$ be a continuous function.

    (a) Show that f is injective if and only if f is strictly monotonic.
    (b) Show that if f is injective then $f : I \to f(I)$ is a homeomorphism.

18. (x^n is a homeomorphism) Show that the function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a homeomorphism. (See [Bou, Top. Ch. IV §3 no. 3]).

23.2 Axioms for $\mathbb{R}_{\geq 0}$ and Dedekind cuts

1. (Ordered sets with addition and Archimedes' property) This is from [Bou, Ch. V §2]. Let E be a totally ordered set with order $\leq$ and smallest element 0. Let

$$I \subseteq E \qquad \text{with a function} \quad \begin{matrix} I \times I & \to & E \\ (x, y) & \to & x + y \end{matrix}$$

such that

(I1) $0 \in I$,
(I2) If $x \in I$ and $y \in E$ and $y \leq x$ then $y \in I$,

and

(GR_I) If $x \in I$ then $x + 0 = 0 + x = x$,
(GR'_I) If $x, y, z \in I$ then $x + (y + z) = (x + y) + z$,
(GR_{II}) If $x, y, z \in I$ and $x < y$ then $x + z < y + z$ and $z + x < z + y$,
(GR_{III}) $\{y \in I \mid y > 0\} \neq \emptyset$ and $\{y \in I \mid y > 0\}$ and has no smallest element,
(GR'_{III}) If $x, y \in I$ and $x < y$ then there exists $z \in I$ such that $z > 0$ and $x + z \leq y$,
(GR_{IV}) ("Archimedes' axiom") If $x, y \in I$ and $x > 0$ then there exists $n \in \mathbb{Z}_{>0}$ such that $nx \in E$ and $nx > y$.

Show that there exists a strictly increasing function $f \colon I \to \mathbb{R}_{\geq 0}$ such that

$$\text{if } x, y \in I \text{ and } x + y \in I \text{ then} \quad f(x + y) = f(x) + f(y).$$

Moreover, if $b \in I$ then $f(I) \cap [0, f(b)]$ is dense in $[0, f(b)]$.

Proof seed: Let $\mathcal{F}$ the filter on I for which the sets $\{[0, z] \mid z \in I, z > 0\}$ form a base. Let $a \in I$ with $a > 0$ and define

$$f(x) = \lim_{z \to 0} \frac{\lfloor \frac{x}{z} \rfloor}{\lfloor \frac{a}{z} \rfloor}, \qquad \text{where} \quad \left\lfloor \frac{x}{z} \right\rfloor = \max\{n \in \mathbb{Z}_{\geq 0} \mid nz \leq x\},$$

and $\lim_{z \to 0}$ is the limit with respect to the filter $\mathcal{F}$ on I.

2. (Replacing Archimedes' property with the least upper bound property) Let E be a totally ordered set with order $\leq$ and smallest element 0. Let

$$I \subseteq E \qquad \text{with a function} \quad \begin{matrix} I \times I & \to & E \\ (x, y) & \to & x + y \end{matrix}$$

such that

(I1) $0 \in I$,
(I2) If $x \in I$ and $y \in E$ and $y \leq x$ then $y \in I$,

and

(GR_I) If $x \in I$ then $x + 0 = 0 + x = x$,
(GR'_I) If $x, y, z \in I$ then $x + (y + z) = (x + y) + z$,
(GR_{II}) If $x, y, z \in I$ and $x < y$ then $x + z < y + z$ and $z + x < z + y$,

Assume

(GR_{IIIa}) $\{y \in I \mid y > 0\} \neq \emptyset$ and $\{y \in I \mid y > 0\}$ and has no smallest element,
(GR'_{IIIa}) If $x, y \in I$ and $x < y$ then there exists $z \in I$ such that $x + z = y$,
(GR_{IVa}) Every increasing sequence of elements of I which is bounded above by an element of I has a least upper bound in I.

Then

(GR_{IV}) ("Archimedes' axiom") If $x, y \in I$ and $x > 0$ then there exists $n \in \mathbb{Z}_{>0}$ such that $nx \in E$ and $nx > y$;

holds. (See [Bou, Top. Ch. V §2].)

3. (Building the correspondence to $\mathbb{R}_{\geq 0}$) This is from [Bou, Ch. V §2]. Let E be a totally ordered set with order $\leq$ and smallest element 0. Let

$$I \subseteq E \qquad \text{with a function} \quad \begin{matrix} I \times I & \to & E \\ (x, y) & \to & x + y \end{matrix}$$

such that

(I1) $0 \in I$,
(I2) If $x \in I$ and $y \leq x$ then $y \in I$,

and

(GR_I) If $x \in I$ then $x + 0 = 0 + x = x$,
(GR'_I) If $x, y, z \in I$ then $x + (y + z) = (x + y) + z$,
(GR_{II}) If $x, y, z \in I$ and $x < y$ then $x + z < y + z$ and $z + x < z + y$,

Assume

(GR_{IIIa}) $\{y \in I \mid y > 0\} \neq \emptyset$ and $\{y \in I \mid y > 0\}$ and has no smallest element,
(GR'_{IIIa}) If $x, y \in I$ and $x < y$ then there exists $z \in I$ such that $x + z = y$,
(GR_{IVa}) Every increasing sequence of elements of I which is bounded above by an element of I has a least upper bound in I.

Show that there exists $z \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and a strictly increasing surjective function $f\colon I \to [0, z)$ or $f\colon I \to [0, z]$ such that $f(0) = 0$ and

$$\text{if } x, y \in I \text{ and } x + y \in I \text{ then} \quad f(x + y) = f(x) + f(y).$$

4. (Dedekind cuts) A *cut* is a subset x of $\mathbb{Q}_{>0}$ such that

   (a) $x \neq \emptyset$ and $x \neq \mathbb{Q}_{>0}$,
   (b) (lower ideal) If $p \in x$ and $q \in \mathbb{Q}_{>0}$ and $q < p$ then $q \in x$,
   (c) (no maximal element) If $p \in x$ then there exists $r \in x$ with $p < r$.

Define a totally ordered set

$$\mathbb{R}_{>0} = \{\text{cuts in } \mathbb{Q}_{\geq 0}\}, \qquad \text{with} \quad x \leq y \quad \text{if} \quad x \subseteq y.$$

Define an addition on $\mathbb{R}_{>0}$ by

$$x + y = \{r + s \mid r \in x \text{ and } s \in y\},$$

and define a multiplication on $\mathbb{R}_{>0}$ by

$$xy = \{p \in \mathbb{Q}_{>0} \mid \text{there exists } r \in x \text{ and } s \in y \text{ with } p < rs\}.$$

Finally, define

$$\iota\colon \mathbb{Q}_{>0} \to \mathbb{R}_{>0} \qquad \text{by} \qquad \iota(r) = \{p \in \mathbb{Q}_{>0} \mid p < r\}.$$

Show that

(1) ι is injective,
(2) $\mathbb{R}_{>0}$ has no smallest element,
(3) If $x, y, z \in \mathbb{R}_{>0}$ then $(x + y) + z = x + (y + z)$,
(4) If $x, y, z \in \mathbb{R}_{>0}$ and $x \leq y$ then $x + z \leq y + z$,
(5) If $x, y \in \mathbb{R}_{>0}$ and $x < y$ then there exists $z \in \mathbb{R}_{>0}$ such that $x + z = y$,
(6) If $E \subseteq \mathbb{R}_{>0}$ and $E \neq \emptyset$ and E is bounded then $\sup(E)$ exists.

(see [BRu, Ch. 1 Appendix]).

23.3 The mean value theorem

1. (continuous images of connected sets are connected and continuous images of compact sets are compact) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The set E is *connected* if there do not exist open sets A and B in X ($A, B \in \mathcal{T}$) with

$$A \cap E \neq \emptyset \quad \text{and} \quad B \cap E \neq \emptyset \quad \text{and} \quad A \cup B \supseteq E \quad \text{and} \quad (A \cap B) \cap E = \emptyset.$$

The set E is *compact* if E satisfies

$$\begin{aligned} &\text{if } \mathcal{S} \subseteq \mathcal{T} \text{ and } E \subseteq \left(\bigcup_{U \in \mathcal{S}} U\right) \quad \text{then there exists} \\ &\ell \in \mathbb{Z}_{>0} \text{ and } U_1, U_2, \ldots, U_\ell \in \mathcal{S} \text{ such that } E \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell. \end{aligned}$$

Let $f\colon X \to Y$ be a continuous function and let $E \subseteq X$. Show that

(a) If E is connected then $f(E)$ is connected,
(b) If E is compact then $f(E)$ is compact.

2. (connected subsets of $\mathbb{R}$ are intervals) Let $A \subseteq \mathbb{R}$, where the metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$. Show that

$$A \text{ is connected} \quad \text{if and only if} \quad A \text{ is an interval},$$

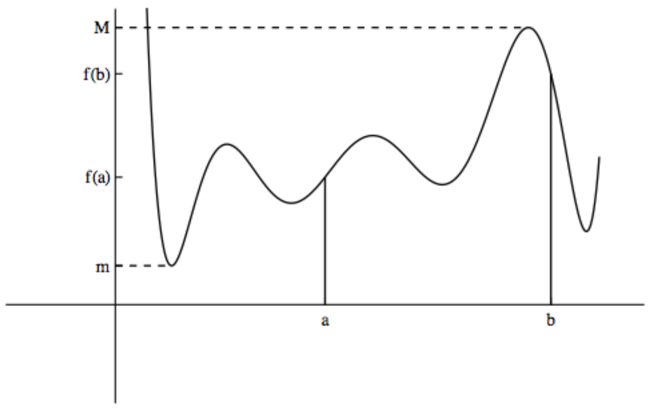
i.e. A is connected if and only if there exist $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$ such that $A = (a, b)$ or $A = [a, b)$ or $A = (a, b]$ or $A = [a, b]$.

3. (connected compact subsets of $\mathbb{R}$ are closed bounded intervals) Let $A \subseteq \mathbb{R}$, where the metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$. Show that

$$A \text{ is connected and compact} \quad \text{if and only if} \quad A \text{ is a closed bounded interval,}$$

i.e. A is connected and compact if and only if there exist $a, b \in \mathbb{R}$ such that $A = [a, b]$.

4. ($f: [a, b] \to \mathbb{R}$ have minimums and maximums and intermediate values) Let $a, b \in \mathbb{R}$ with $a < b$.
   (a) Show that if $f: [a, b] \to \mathbb{R}$ is a continuous function and $w \in (f(a), f(b))$ then there exists $c \in (a, b)$ such that $f(c) = w$.
   (b) Show that if $f: [a, b] \to \mathbb{R}$ is a continuous function then there exist $m, M \in \mathbb{R}$ such that $f([a, b]) = [m, M]$.



The intermediate value theorem.

5. (Rolle's theorem) Let $a, b \in \mathbb{R}$ with $a < b$. Show that if $f: [a, b] \to \mathbb{R}$ is a function such that $f: [a, b] \to \mathbb{R}$ is continuous and $f': (a, b) \to \mathbb{R}$ exists

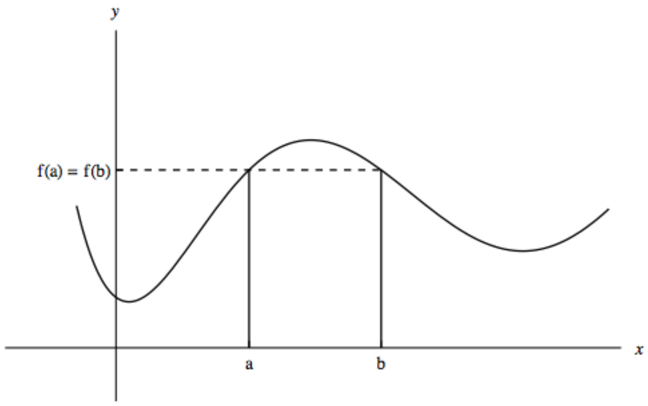
$$\text{and } f(a) = f(b) \qquad \text{then there exists } c \in (a, b) \text{ such that} \qquad f'(c) = 0.$$

6. (The mean value theorem) Let $a, b \in \mathbb{R}$ with $a < b$. Show that if $f: [a, b] \to \mathbb{R}$ is a function such that $f: [a, b] \to \mathbb{R}$ is continuous and $f': (a, b) \to \mathbb{R}$ exists then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

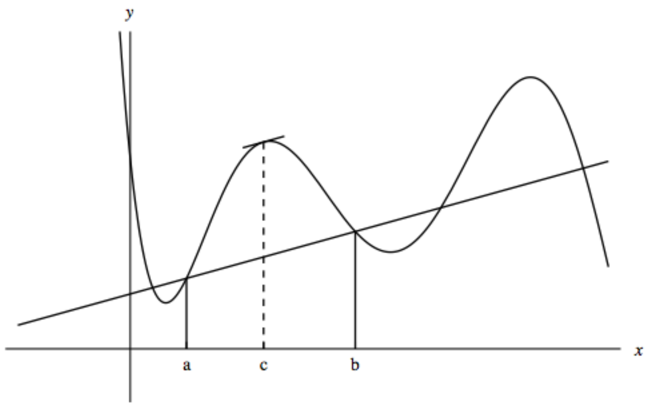
7. (Taylor's theorem) Let $a, b \in \mathbb{R}$ with $a < b$ and $N \in \mathbb{Z}_{\geq 0}$. Show that if $f: [a, b] \to \mathbb{R}$ is a function such that $f^{(N)}: [a, b] \to \mathbb{R}$ is continuous and $f^{(N+1)}: (a, b) \to \mathbb{R}$ exists then there exists $c \in (a, b)$ such that

$$\begin{aligned} f(b) = f(a) + f'(a)(b - a) + \frac{1}{2!}f''(a)(b - a)^2 + \cdots \\ + \frac{1}{N!}f^{(N)}(a)(b - a)^N + \frac{1}{(N + 1)!}f^{(N+1)}(c)(b - a)^{N+1}. \end{aligned}$$



Rolle's theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



The mean value theorem

23.4 Properties of $\mathbb{Q}_p$

1. (reducing to primes p) (See [Mah, Ch. 5].)

   (a) Let $p \in \mathbb{Z}_{>1}$ be prime and let $r \in \mathbb{Z}_{>0}$. Show that $\mathbb{Q}_{p^r} \cong \mathbb{Q}_p$.
   (b) Let $\ell, m \in \mathbb{Z}_{>0}$ with $\gcd(\ell, m) = 1$. Show that $\mathbb{Q}_{mn} = \mathbb{Q}_\ell \oplus \mathbb{Q}_m$.
   (c) Let $\ell \in \mathbb{Z}_{>0}$ and let $\ell = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the prime factorization of ℓ. Show that $\mathbb{Q}_\ell \cong \mathbb{Q}_{p_1} \oplus \mathbb{Q}_{p_2} \oplus \cdots \oplus \mathbb{Q}_{p_k}$.

2. ($\mathbb{Q}_p$ is a field if p is prime) Let $p \in \mathbb{Z}_{>1}$ be prime. Show that $\mathbb{Q}_p$ is a field if and only if p is prime.

23.5 Triangle inequalities

1. (Cauchy-Schwarz and triangle inequalities in $\mathbb{R}^n$) Let $x, y \in \mathbb{R}^n$. Prove the following:

   (a) (Lagrange's identity) $|x|^2 \cdot |y|^2 - \langle x, y\rangle^2 = \frac{1}{2}\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2$.
   (b) (Cauchy-Schwarz inequality) $\langle x, y\rangle \le |x| \cdot |y|$.
   (c) (triangle inequality) $|x + y| \le |x| + |y|$.

2. (Cauchy-Schwarz and triangle inequalities in inner product spaces) Let $(V, \langle,\rangle)$ be a positive definite inner product space.

   (a) (Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.
   (b) (triangle inequality) Showthat if $x, y \in V$ then $\|x + y\| \le \|x\| + \|y\|$.

3. (Hölder and Minkowski inequalities) Let $q \in \mathbb{R}_{\ge 1}$ and let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$.

   (a) (Young's inequality) Show that if $a, b \in \mathbb{R}_{>0}$ then $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{1}{p} a + \frac{1}{q} b$.
   (b) (Hölder inequality for $\mathbb{R}^n$) Show that if $x, y \in \mathbb{R}^n$ then $|\langle x, y\rangle| \le \|x\|_p \|y\|_q$.
   (c) (Minkowski inequality for $\mathbb{R}^n$) Show that if $x, y \in \mathbb{R}^n$ then $\|x + y\|_p \le \|x\|_p + \|y\|_p$.
   (d) (Hölder inequality) Show that if $x \in \ell^p$ and $y \in \ell^q$ then $|\langle x, y\rangle| \le \|x\|_p \|y\|_q$.
   (e) (Minkowski inequality) Show that if $x \in \ell^p$ and $y \in \ell^q$ then $\|x + y\|_p \le \|x\|_p + \|y\|_p$.

23.6 The spaces ℓ^p

1. (Containment of ℓ^p-spaces) [Bressan, Ch. 2 Ex. 14] Let $p, s \in \mathbb{R}_{>1}$. Show that if $p \le s$ then $\ell^p \subseteq \ell^s$.

2. (ℓ^p-spaces depend on p) [Bressan, Ch. 2 Ex. 14] Let $p, s \in \mathbb{R}_{>1}$. Show that if $p \ne s$ then $\ell^p \ne \ell^s$.

3. (the dual of $\mathbb{R}^2$ in the $\|\ \|_p$ norm) [Bressan, Ch. 2 Ex. 21] Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional, say $\phi(x_1, x_2) = ax_2 + bx_2$. Give a direct proof that

   (a) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_1 = |x_1| + |x_2|$ then the corresponding operator norm is $\|\phi\|_\infty = \max\{|a|, |b|\}$.
   (b) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ then the corresponding operator norm is $\|\phi\|_1 = |a| + |b|$.
   (b) If $p \in \mathbb{R}_{>1}$ and $\mathbb{R}^2$ is endowed with the norm $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ then the corresponding operator norm is $\|\phi\|_p = (|a|^q + |b|^q)^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

4. (Dual of an ℓ^p-space) Let $p \in \mathbb{R}_{>1}$. Show that $(\ell^p)^* = \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

5. (Dual of c_0) Show that $(c_0)^* = \ell^1$. (See Rudin, Real and complex analysis, Ch. 5 Ex. 9)

6. (Dual of ℓ^1) Show that $(\ell^1)^* = \ell^\infty$. (See Rudin, Real and complex analysis, Ch. 5 Ex. 9)

7. (Dual of ℓ^∞) [Bressan, Ch. 2 Ex. 27] Show that $(\ell^\infty)^* \ne \ell^1$. (See Rudin, Real and complex analysis, Ch. 5 Ex. 9)

8. (ℓ^p is complete) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is a complete metric space.

9. (ℓ^1 is complete) Show that ℓ^1 is a complete metric space.

10. (ℓ^∞ is complete) Show that ℓ^∞ is a complete metric space.

11. (c_0 is complete) Show that c_0 is the completion of
$$c_c = \{(x_1, x_2, \ldots) \in \ell^\infty \ \text{ all but a finite number of } x_i \text{ are } 0\}$$
the space of sequences that are eventually 0. (IS THE esssup NORM AND THE SUP NORM THE SAME FOR COUNTING MEASURE? SEE Theorem 1.3 on the page http://www.ms.unimelb.edu.a

12. (The completion of c_c with respect to $\|\ \|_p$) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is the completion of
$$c_c = \{(x_1, x_2, \ldots) \in \ell^p \ \text{ all but a finite number of } x_i \text{ are } 0\}$$
the space of sequences that are eventually 0.

13. (the closure of the span of the standard basis in ℓ^p) [Bressan, Ch. 2 Ex. 15] Let
$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots,$$
and
$$\text{let } p \in \mathbb{R}_{>1}. \qquad \text{Show that,} \quad \text{in } \ell^p, \qquad \overline{\text{span}\{e_1, e_2, \ldots\}} = \ell^p.$$

14. (the closure of the span of the standard basis in ℓ^1) [Bressan, Ch. 2 Ex. 15] Let
$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots.$$
Show that,
$$\text{in } \ell^1, \qquad \overline{\text{span}\{e_1, e_2, \ldots\}} = \ell^1.$$

15. (the closure of the span of the standard basis in ℓ^∞) [Bressan, Ch. 2 Ex. 15] Let
$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots.$$
Show that,
$$\text{in } \ell^\infty, \qquad \overline{\text{span}\{e_1, e_2, \ldots\}} = c_0.$$

16. (weak convergence of of the standard basis in ℓ^p) [Bressan, Ch. 2 Ex. 36] Let
$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots,$$
and let $p \in \mathbb{R}_{>1}$. Show that
$$\text{in } \ell^p, \quad \text{the sequence } (e_1, e_2, e_3, \ldots) \text{ weakly converges weakly to } 0.$$

17. (weak convergence of of the standard basis in ℓ^p) [Bressan, Ch. 2 Ex. 36] Let
$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots,$$
Show that,
$$\text{in } \ell^1, \ (e_1, e_2, e_3, \ldots) \text{ does not have any weakly convergent subsequence.}$$

23.7 Absolute values and valuations

1. (absolute values and norms) [from Pete Clark, lecture notes on valuation theory] Let $\mathbb{F}$ be a field. A *norm* on $\mathbb{F}$ is a function $|\ |: \mathbb{F} \to \mathbb{R}_{\geq 0}$ such that

   (a) if $x \in \mathbb{F}$ then $|x| = 0$ if and only if $x = 0$,
   (b) if $x, y \in \mathbb{F}$ then $|xy| = |x||y|$,
   (c) if $x, y \in \mathbb{F}$ then $|x + y| \leq |x| + |y|$.

   An *absolute value on* $\mathbb{F}$ is a function $|\ |: \mathbb{F} \to \mathbb{R}_{\geq 0}$ such that

   (a) if $x \in \mathbb{F}$ then $|x| = 0$ if and only if $x = 0$,
   (b) if $x, y \in \mathbb{F}$ then $|xy| = |x||y|$,
   (c) There exists $C \in \mathbb{R}_{>0}$ such that if $x \in \mathbb{F}$ and $|x| \leq 1$ then $|x + 1| \leq C$.

   Show that this last condition is equivalent to

   $$\text{if } x, y \in \mathbb{F} \quad \text{then} \quad |x + y| \leq C\max\{|x|, |y|\}.$$

   Show that $C \geq 1$. A *non-Archimedean absolute value* is an absolute value on $\mathbb{F}$ such that $C = 1$. Show that an absolute value on $\mathbb{F}$ is a norm on $\mathbb{F}$ if and only if $C \leq 2$. Show that if $|\ |$ is an absolute value on $\mathbb{F}$ with constant C (take the inf to make this unique) and $\alpha \in \mathbb{R}_{>0}$ then $|\ |^\alpha$ is an absolute value on $\mathbb{F}$ with constant C^α.

2. (valuations are the logs of absolute values) Let $(\mathbb{F}, |\ |)$ be a non-Archimedean normed field. The *valuation ring* is

   $$\mathfrak{o} = \{x \in \mathbb{F} \mid |x| \leq 1\} \qquad \mathfrak{m} = \{x \in \mathbb{F} \mid |x| < 1\}$$

   is the unique maximal ideal of $\mathfrak{o}$ WHAT IS THE STATEMENT???

3. (valuation rings give valuations) [Atiyah-Macondald, Ch. 5 Ex. 30] Let $\mathbb{F}$ be a field. A *valuation ring of* $\mathbb{F}$ is a subring ring $\mathfrak{o} \subseteq \mathbb{F}$ such that

   (a) $\mathfrak{o}$ is an integral domain,
   (b) $\mathbb{F}$ is the field of fractions of $\mathfrak{o}$,
   (c) If $x \in \mathbb{F}^\times$ then $x \in \mathfrak{o}$ or $x^{-1} \in \mathfrak{o}$.

   Let $\mathfrak{o}^\times = \{a \in \mathfrak{o} \mid a \text{ is invertible in } \mathfrak{o}\}$, let

   $$\Gamma = \frac{\mathbb{F}^\times}{\mathfrak{o}^\times}, \qquad \text{and let} \quad \mathrm{val}\colon \mathbb{F}^\times \to \Gamma$$

   be the quotient map. Define a partial order $\leq$ on Γ by

   $$a \leq b \qquad \text{if} \qquad ab^{-1} \in \mathfrak{o}.$$

   Show that Γ is a totally ordered abelian group and val is a valuation of $\mathbb{F}$ with values in Γ.

4. (valuations give valuation rings) [Atiyah-Macdonald, Ch. 5 Ex. 31] Let Γ be a totally ordered abelian group. Let $\mathbb{F}$ be a field. A *valuation of* $\mathbb{F}$ *with values in* Γ is a function

   $$\mathrm{val}\colon \mathbb{F}^\times \to \Gamma \qquad \text{such that}$$

(a) if $x, y \in \mathbb{F}^\times$ then $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$,
(b) if $x, y \in \mathbb{F}^\times$ then $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$,

ARE THESE TWO INEQUALITIES IN THE CORRECT DIRECTION? The *valuation ring of* val is

$$\mathfrak{o} = \{x \in \mathbb{F}^\times \mid \mathrm{val}(x) \geq 0\}.$$

Show that $\mathfrak{o}$ is a valuation ring of $\mathbb{F}$.

23.8 Additional sample exam questions

23.8.1 Favourite examples

1. (Definition of the nonnegative real numbers)

   (a) Carefully define the nonnegative real numbers $\mathbb{R}_{\geq 0}$.
   (b) Carefully define the usual addition and multiplication on $\mathbb{R}_{\geq 0}$.
   (c) Carefully define the usual order on $\mathbb{R}_{\geq 0}$.
   (d) Carefully define the usual topology $\mathbb{R}_{\geq 0}$.

   Be careful that your definitions are not circular (i.e. be careful that your definitions are not somehow already using the real numbers to define the real numbers).

2. (Properties of the order on $\mathbb{R}_{\geq 0}$)

   (a) Prove that if $a, b, c \in \mathbb{R}_{\geq 0}$ and $a \leq b$ then $a + c \leq b + c$.
   (b) Prove that if $x, y \in \mathbb{R}_{\geq 0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.
   (c) Prove that if $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$ then there exists $c \in \mathbb{Q}_{\geq 0}$ (a rational number) such that $a < c < b$.
   (d) Prove that if $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$ then there exists $c \in (\mathbb{R}_{\geq 0} - \mathbb{Q}_{\geq 0})$ (an irrational number) such that $a < c < b$.

3. (Least upper bounds and increasing sequences in $\mathbb{R}_{\geq 0}$)

   (a) Prove that if $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists.
   (b) Give an example (with proof) of an increasing sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.
   (c) Give an example (with proof) of a bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.
   (d) Prove that if $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, a_3, \ldots)$ converges.
   (e) Give an example (with proof) of an increasing and bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\geq 0}$ which does not converge.

4. (Properties of the topology on $\mathbb{R}_{\geq 0}$)

   (a) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$. Prove that (a, b) is open in $\mathbb{R}_{\geq 0}$.
   (b) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$. Prove that $[a, b]$ is closed in $\mathbb{R}_{\geq 0}$.
   (c) Define compact and prove that $\mathbb{R}_{\geq 0}$ is not compact.

(d) Define locally compact and prove that $\mathbb{R}_{\ge 0}$ is locally compact.

5. (Properties of the uniform structure on $\mathbb{R}_{\ge 0}$)

   (a) Carefully define the usual uniformity on $\mathbb{R}_{\ge 0}$.
   (b) Define complete and prove that $\mathbb{R}_{\ge 0}$ is complete.

6. (Connected and compact subsets of $\mathbb{R}_{\ge 0}$) Let $A \subseteq \mathbb{R}_{\ge 0}$.

   (a) Prove that A is connected if and only if A is an interval.
   (b) Prove that A is compact if and only if A is closed and bounded.

7. Assume that it is known that $\mathbb{R}_{\ge 0}$ is complete.

   (a) Prove that if $A \subseteq \mathbb{R}_{\ge 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists.
   (b) Give an example (with proof) of an increasing sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\ge 0}$ which does not converge.
   (c) Give an example (with proof) of a bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\ge 0}$ which does not converge.
   (d) Prove that if $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\ge 0}$ then $(a_1, a_2, a_3, \ldots)$ converges.
   (e) Give an example (with proof) of an increasing and bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\ge 0}$ which does not converge.

8. Let $X = \{0, 1\}$ and let $\mathcal{T} = \{\emptyset, X, \{0\}\}$.

   (a) Show that $\mathcal{T}$ is a topology on X.
   (b) Show that there does not exist a metric $d\colon X \times X \to \mathbb{R}_{\ge 0}$ such that $\mathcal{T}$ is the metric space topology of (X, d).

9. Define the standard metric on $\mathbb{C}$ and show that $\mathbb{C}$, with this metric, is a metric space.

10. Let d be the standard metric on $\mathbb{C}$. Show that $\mathbb{R}$ is a metric subspace of $(\mathbb{C}, d)$.

11. Define the standard metric on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this metric, is a metric space.

12. Define the standard norm on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this norm, is a normed vector space.

13. Define the norm $\| \, \|_p$ on $\mathbb{R}^n$ and show that $(\mathbb{R}^n, \| \, \|_p)$ is a normed vector space.

14. Let X be a nonempty set. Define the set of bounded functions $B(X, \mathbb{R})$ and the sup norm on $B(X, \mathbb{R})$. Show that $B(X, \mathbb{R})$, with this norm, is a normed vector space.

15. Let $a, b \in \mathbb{R}$ with $a < b$. Define the set of continuous functions $C([a,b],\mathbb{R})$ and the L^1-norm on $C([a,b],\mathbb{R})$. Show that $C([a,b],\mathbb{R})$, with this norm, is a normed vector space.

16. Let $a, b \in \mathbb{R}$ with $a < b$. Show that the set $C_{\mathrm{bd}}([a,b]),\mathbb{R})$ of bounded continuous functions is a metric subspace of $C([a,b],\mathbb{R})$ with the L^1-norm.

17. Let (X,d) and (Y,d') be metric spaces and let $C_b(X,Y)$ be the set of bounded continuous functions $f\colon X \to Y$ with the metric $\rho\colon C_b(X,Y)\times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by
$$\rho(f,g) = \sup\{d'(f(x),g(x)) \mid x \in X\}.$$
Show that if (Y,d') is complete then $(C_b(X,Y),\rho)$ is a complete metric space.

18. Let (X,d) and (Y,d') be metric spaces and let $C_b(X,Y)$ be the set of bounded continuous functions $f\colon X \to Y$ with the metric $\rho\colon C_b(X,Y)\times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by
$$\rho(f,g) = \sup\{d'(f(x),g(x)) \mid x \in X\}.$$
Show that $(C_b(X,Y),\rho)$ is a metric space.

19. We can define d on $(1,\infty) \subseteq \mathbb{R}$ by
$$d(x,y) = \begin{cases} |\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}| & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$
   (a) Explain briefly why d is a metric on $(1,\infty)$
   (b) Show that the mapping $\phi : (1,\infty) \to (0,1)$ defined by $\phi(x) = \frac{1}{\sqrt{x}}$ is an isometry from the metric space $((1,\infty),d)$ to $(0,1)$ with the usual Euclidean metric.
   (c) Is the metric space $((1,\infty),d)$ connected?
   (d) Is the metric space $((1,\infty),d)$ compact?
   Give brief explanations.

20. Let $p \in \mathbb{R}_{>1}$ and define $q \in \mathbb{R}_{>1}$ by $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) Define the normed vector space ℓ^p.
   (b) Show that ℓ^p is a Banach space.
   (c) Prove that the dual of ℓ^p is ℓ^q.

21. Let $p \in \mathbb{R}_{\geq 1}$ and define
$$\ell^p = \{(x_1,x_2,\ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\}, \quad \text{where} \quad \|\vec{x}\|_p = \left(\sum_{i\in\mathbb{Z}_{>0}} |x_i|^p\right)^{1/p}$$
for a sequence $\vec{x} = (x_1,x_2,\ldots) \in \mathbb{R}^\infty$.
   (a) Show that if $p \leq q$ then $\ell^p \subseteq \ell^q$.

(b) Show that if $p \neq q$ then $\ell^p \neq \ell^q$.

22. Let $p \in \mathbb{R}_{>1}$. Let $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis of ℓ^p.

23. Check if the following functions are metrics on X.

    (a) $d(x, y) = |x^2 - y^2|$ for $x, y \in X = \mathbb{R}$
    (b) $d(x, y) = |x^2 - y^2|$ for $x, y \in X = (-\infty, 0]$
    (c) $d(x, y) = |\arctan x - \arctan y|$ for $x, y \in X = \mathbb{R}$

24. (**French railroad metric**) Let $X = \mathbb{R}^2$ and let d be the usual metric. Denote by $\mathbf{0} = (0, 0)$ and define
$$d_{\mathbf{0}}(x, y) = \begin{cases} 0, & \text{if } x = y; \\ d(x, \mathbf{0}) + d(\mathbf{0}, y), & \text{if } x \neq y. \end{cases}$$
Verify that d_0 is a metric on X. (Paris is at the origin $\mathbf{0}$.)

25. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ define
$$d(x, y) = \begin{cases} 1/2 & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or if } x_1 \neq y_1 \text{ and } x_2 = y_2; \\ 1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2; \\ 0 & \text{otherwise.} \end{cases}$$
Verify that d is a metric and that two congruent rectangles, one with base parallel to the x-axis and the other at 45^o to the x-axis, have different "area" if d is used to measure the length of sides.

26. Let (X, d) be a metric space. Consider the function $f : [0, \infty) \to [0, \infty)$ having the following properties:

    (a) f is non-decreasing, i.e. $f(a) \leq f(b)$ if $0 \leq a < b$;
    (b) $f(x) = 0$ if and only if $x = 0$;
    (c) $f(a + b) \leq f(a) + f(b)$, $a, b \in [0, \infty)$.

    If $x, y \in X$ define $d_f(x, y) = f(d(x, y))$. Show that d_f is a metric and that the functions $f(t) = kt$ where $k > 0$, $f(t) = t^\alpha$ where $0 < \alpha \leq 1$, and $f(t) = \frac{t}{1+t}$ for $t \geq 0$ have properties (a)–(c).

27. (**p-adic metric**) Let p be a prime number. Define the p-*adic absolute value function* $|\ |_p$ on $\mathbb{Q}$ by setting $|x|_p = 0$ when $x = 0$ and $|x|_p = p^{-k}$ when $x = p^k \cdot \frac{m}{n}$ where m, n are nonzero integers which are not divisible by p. Show that for $x, y \in \mathbb{Q}$,
$$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$
and that $d(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$. In fact, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. If d satisfies this condition which is stronger than the triangle inequality then d is called an *ultrametric*.

28. Sketch the open ball $B(0,1)$ in the metric space $(\mathbb{R}^3, d_i)$, where d_i is defined by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$
$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$
$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}.$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

29. Set
$$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$$
for $n, m \in \mathbb{Z}_{>0}$. Then d is a metric.

   (a) Let $P \subset \mathbb{Z}_{>0}$ be the set of positive even numbers. Find $\mathrm{diam}(P)$ and $\mathrm{diam}(\mathbb{Z}_{>0} \setminus P)$ in $(\mathbb{Z}_{>0}, d)$.
   (b) For a fixed $n \in \mathbb{Z}_{>0}$ find all elements of $B(2n, \frac{1}{2n})$ and $B(n, \frac{1}{2n})$.

30. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X$ define
$$d_M(x,y) = \begin{cases} |x_2 - y_2|, & \text{if } x_1 = y_1, \\ |x_1 - y_1| + |x_2| + |y_2|, & \text{if } x_1 \neq y_1. \end{cases}$$
Also define
$$d_K(x,y) = \begin{cases} \|x - y\| & \text{if } x = ty \text{ for some } t \in \mathbb{R}; \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$
where $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. (Can you give reasonable interpretations of the metrics d_M and d_K?)

   Study the convergence of the sequence x_n in the spaces (X, d_M) and (X, d_K) if

   (a) $x_n = (\frac{1}{n}, \frac{n}{n+1})$;
   (b) $x_n = (\frac{n}{n+1}, \frac{n}{n+1})$;
   (c) $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$.

23.8.2 Open, closed, dense and nowhere dense sets

1. Give two metrics d and d' on $\mathbb{R}$ such that $\mathbb{Q}$ is open in the metric space topology on $(\mathbb{R}, d)$ and $\mathbb{Q}$ is not open in the metric space topology on $(\mathbb{R}, d')$.

2. In $\mathbb{R}$ with the usual topology give an example of

   (a) a set $A \subseteq \mathbb{R}$ which is both open and closed,
   (b) a set $B \subseteq \mathbb{R}$ which is open and not closed,
   (c) a set $C \subseteq \mathbb{R}$ which is closed and not open,
   (d) a set $D \subseteq \mathbb{R}$ which is not open and not closed.

3. Let $X = \mathbb{R}$ with the usual topology. Show that
   (a) $[0, 1) \subseteq \mathbb{R}$ is not open and not closed,
   (b) $\mathbb{Q} \subseteq \mathbb{R}$ is not open and not closed.

4. Consider the set $X = [-1, 1]$ as a metric subspace of $\mathbb{R}$ with the standard metric. Let
   (a) $A = \{x \in X \mid 1/2 < |x| < 2\}$;
   (b) $B = \{x \in X \mid 1/2 < |x| \leq 2\}$;
   (c) $C = \{x \in \mathbb{R} \mid 1/2 \leq |x| < 1\}$;
   (d) $D = \{x \in \mathbb{R} \mid 1/2 \leq |x| \leq 1\}$;
   (e) $E = \{x \in \mathbb{R} \mid 0 < |x| \leq 1 \text{ and } 1/x \notin \mathbb{Z}\}$.

   Classify the sets in (a)–(e) as open/closed in X and $\mathbb{R}$.

5. Consider $\mathbb{R}^2$ with the standard metric. Let
   (a) $A = \{(x, y) | -1 < x \leq 1 \text{ and } -1 < y < 1\}$;
   (b) $B = \{(x, y) | \, xy = 0\}$;
   (c) $C = \{(x, y) | \, x \in \mathbb{Q}, y \in \mathbb{R}\}$;
   (d) $D = \{(x, y) | -1 < x < 1 \text{ and } y = 0\}$;
   (e) $E = \bigcup_{n=1}^{\infty} \{(x, y) | x = 1/n \text{ and } |y| \leq n\}$.

   Sketch (if possible) and classify the sets in (a)–(e) as open/closed/neither in $\mathbb{R}^2$.

6. Find the interior, the closure and the boundary of each of the following subsets of $\mathbb{R}^2$ with the standard metric:
   (a) $A = \{(x, y)) \mid x > 0 \text{ and } y \neq 0\}$;
   (b) $B = \{(x, y) \mid x \in \mathbb{Z}_{>0}, y \in \mathbb{R}\}$;
   (c) $C = A \cup B$;
   (d) $D = \{(x, y) \mid x \text{ is rational}\}$;
   (e) $F = \{(x, y) \mid x \neq 0 \text{ and } y \leq 1/x\}$.

7. Let $X = \mathbb{R}$ with the usual topology.
   (a) Determine (with proof) $\partial([0, 1])$.
   (b) Determine $\partial\mathbb{Q}$ (with proof, of course).

8. Let $X = \mathbb{R}$ with the usual topology. Show that
   (a) $\mathbb{Z}_{>0}$ is a discrete set in $\mathbb{R}$.
   (b) $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\} \subseteq \mathbb{R}$ is a discrete set in $\mathbb{R}$.

9. Let $X = \mathbb{R}$ with the usual topology.

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
(b) Show that $\mathbb{Q}^c$ is dense in $\mathbb{R}$.
(c) Show that $\mathbb{Z}_{>0}$ is nowhere dense in $\mathbb{R}$.
(d) Show that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.
(e) Show that $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.

10. Let C be the Cantor set in $\mathbb{R}$, where $\mathbb{R}$ has the usual topology.

(a) Show that C is closed in $\mathbb{R}$.
(b) Show that C does not contain any interval in $\mathbb{R}$.
(c) Show that C has nonempty interior.
(d) Show that C is nowhere dense in $\mathbb{R}$.

11. Let C be the Cantor set and let $Q = \{x \in \mathbb{Q} \mid 0 \le x \le 1\}$. Let C and Q have the subspace topology of the interval $X = [0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ in $\mathbb{R}$, where $\mathbb{R}$ has the standard topology.

(a) Show that C is closed in X and not open in X, and Q is not closed in X and Q is not open in X.
(b) Show that C is nowhere dense in X and Q is dense in X.
(c) Show that C^c is dense in X and Q^c is dense in X.
(d) Show that C is compact and Q is not compact.
(e) Show that C and Q are both totally disconnected (i.e. every connected component is a set with a single point).
(e) Let μ be a function which assigns values to certain subsets of X which satisfies

$$\mu([a,b]) = b - a, \quad \text{if } a, b \in \mathbb{R} \text{ and } 0 \le a < b \le 1,$$

and

$$\mu\Big(\bigcup_{i\in\mathbb{Z}_{>0}} A_i\Big) = \sum_{i\in\mathbb{Z}_{>0}} \mu(A_i) \quad \text{if } A_1, A_2, \ldots \text{ are disjoint subsets of } X\ .$$

Show that

$$\mu(C) = 0, \quad \mu(C^c) = 1, \quad \mu(Q) = 0, \quad \text{and} \quad \mu(Q^c) = 1.$$

(f) Show that $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$, $\mathrm{Card}(C^c) = \mathrm{Card}(\mathbb{R})$, $\mathrm{Card}(Q) \ne \mathrm{Card}(\mathbb{R})$ and $\mathrm{Card}(Q^c) = \mathrm{Card}(\mathbb{R})$.

12. Let $X = \mathbb{R}$ with the usual metric and let $U = \mathbb{Q}$ and $V = \mathbb{Q}^c$. Show that U and V are dense and $U \cap B = \emptyset$.

13. Let $X = \mathbb{Q}$ with the usual metric and let $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{Z}_{>0}$ let $Q_n = \mathbb{Q} - \{q_n\}$.

(a) Show that if $n \in \mathbb{Z}_{>0}$ then Q_n is open and dense.
(b) Show that $\displaystyle\bigcap_{n\in\mathbb{Z}_{>0}} Q_n = \emptyset$.

14. (Ch 3 Neighborhoods etc) Show that $\mathbb{R}$, with the standard topology, cannot be be written as a countable union of nowhere dense sets.

15. Let $X = \mathbb{Q}$, with the standard topology. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q}$. Show that $\{q_n\}$ is nowhere dense. Determine the interior of $\bigcup_{n\in\mathbb{Z}_{>0}} \{q_n\}$.

16. Let (X, d) be a complete metric space and let $(f_1, f_2, f_3, \ldots)$ be a sequence of continuous functions
$$f_n\colon X \to \mathbb{R}, \qquad \text{for } n \in \mathbb{Z}_{>0}.$$
Assume that if $x \in X$ then $(f_1(x), f_2(x), \ldots)$ is bounded in X. Show that there exists an open set $U \subseteq X$ such that
$$\text{there exists } M \in \mathbb{R}_{>0} \quad \text{such that} \quad \text{if } x \in U \text{ and } n \in \mathbb{Z}_{>0} \text{ then } |f_n(x)| \le M.$$

17. Let X be a complete normed vector space over $\mathbb{R}$. A **sphere** in X is a set
$$S(a, r) = \{x \in X : d(x, a) = \|x - a\| = r\}, \qquad \text{for } a \in X \text{ and } r \in \mathbb{R}_{>0}.$$
   (a) Show that each sphere in X is nowhere dense.
   (b) Show that there is no sequence of spheres $\{S_n\}$ in X whose union is X.
   (c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.
   (d) Show that the result of (b) does not hold in every complete metric space X.

23.8.3 Continuity

1. Let X be a topological space and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions.
   (a) Show that $f + g$ is continuous.
   (b) Show that $f \cdot g$ is continuous.
   (a) Show that $f - g$ is continuous.
   (d) Show that if g satisfies if $x \in X$ then $g(x) \neq 0$ then f/g is continuous.

2. Let (X, d) be a metric space. Show that $d\colon X \times X \to \mathbb{R}$ is continuous.

3. Let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by
$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$
If $a \in \mathbb{R}$ let $\ell_a\colon \mathbb{R} \to \mathbb{R}$ be given by $\ell_a(y) = f(a, y)$. If $b \in \mathbb{R}$ let $r_b\colon \mathbb{R} \to \mathbb{R}$ be given by $r_b(x) = f(x, b)$.
   (a) Let $a \in \mathbb{R}$. Show that $\ell_a\colon \mathbb{R} \to \mathbb{R}$ is continuous.
   (b) Let $b \in \mathbb{R}$. Show that $r_b\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(c) Show that f is not continuous at $(0,0)$.

4. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x}{1+x^2}$ is uniformly continuous.

5. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, is not uniformly continuous.

6. Let X and Y be topological spaces and assume that Y is Hausdorff. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions.
   (a) Show that the set $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.
   (b) Show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous then
   $$f - g \quad \text{is continuous.}$$
   (c) Show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous then
   $$\{x \in X \mid f(x) < g(x)\} \quad \text{is open.}$$

7. Let C be the circle in $\mathbb{R}^2$ with the centre at $(0, 1/2)$ and radius $1/2$. Let $X = C \setminus \{(0,1)\}$. Define the function $f : \mathbb{R} \to X$ by defining $f(t)$ to be the point at which the line segment from $(t,0)$ to $(0,1)$ intersects X.
   (a) Show that $f : \mathbb{R} \to X$ and $f^{-1} : X \to \mathbb{R}$ are continuous.
   (b) Define for $s, t \in \mathbb{R}$
   $$\rho(s,t) = |f(s) - f(t)|$$
   where $|\ |$ is the standard norm in $\mathbb{R}^2$. Show that ρ defines a metric on $\mathbb{R}$ which is topologically equivalent to the standard metric on $\mathbb{R}$.

8. Let $X = C[0,1]$. Let $F\colon X \to \mathbb{R}$ be defined by $F(f) = f(0)$. Moreover, let $d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}$ and $d_1(f,g) = \int_0^1 |f(x) - g(x)|dx$.
   Is F continuous when X is equipped with (a) the metric d_∞, (b) the metric d_1?

9. Which of the following functions are uniformly continuous?
   (a) $f(x) = \sin x$ on $[0, \infty)$
   (b) $g(x) = \dfrac{1}{1-x}$ on $(0,1)$
   (c) $h(x) = \sqrt{x}$ on $[0, \infty)$
   (d) $k(x) = \sin(1/x)$, on $(0,1)$

10. Suppose that A is a dense subset of a metric space (X,d) and $f\colon A \to \mathbb{R}$ is uniformly continuous. Show that there exists exactly one continuous function $g\colon X \to \mathbb{R}$ satisfying $g(x) = f(x)$ for $x \in A$. (Hint: You may need to use the completeness of $\mathbb{R}$.)

11. Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $x \in S^1$ such that $f(x) = f(-x)$. (Hint: consider the function $g \colon S^1 \to \mathbb{R}$ where $g(x) = f(x) - f(-x)$.)

12. (Functions on $\mathbb{R}_{\geq 0}$)
    (a) Carefully define continuous and uniformly continuous functions.
    (a) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $x^n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.
    (b) Let $n \in \mathbb{Z}_{>1}$. Prove that the function $x^n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not uniformly continuous.
    (b) Let $n \in \{0, 1\}$. Prove that the function $x^n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.
    (c) Prove that the function $e^x \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.

13. Let X be a connected topological space. Let $f \colon X \to \mathbb{R}$ be continuous with $f(X) \subseteq \mathbb{Q}$. Show that f is a constant function.

14. Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let

$$f \colon [0, 2\pi) \to S^1 \qquad \text{be given by} \qquad f(x) = (\cos x, \sin x).$$

    (a) Show that f is continuous.
    (b) Show that f is a bijection.
    (c) Show that $f^{-1} \colon S^1 \to [0, 2\pi)$ is not continuous.
    (d) Why does this not contradict the following statement: *Let X and Y be topological spaces and let $f \colon X \to Y$ be a continuous function. Assume f is a bijection, X is compact and Y is Hausdorff. Then the inverse function $f^{-1} \colon Y \to X$ is continuous.*

15. Let $X = \mathbb{R}_{\geq 0}$ with metric given by $d(x, y) = |x - y|$. Show that the function

$$\begin{array}{ccc} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & x + y \end{array} \quad \text{is uniformly continuous}$$

and the function

$$\begin{array}{ccc} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & xy \end{array} \qquad \text{is continuous but not uniformly continuous.}$$

23.8.4 Sequences of functions

1. Let (X, d) and (Y, ρ) be metric spaces. Let $\{f_k\}$ be a sequence of functions $f_k \colon X \to Y$ and let $f \colon X \to Y$ be a function. Show that $\{f_k\}$ converges uniformly to f if and only if $\sup\{\rho(f_k(x), f(x)) \mid x \in X\} \to 0$.

2. Let $\{f_k\}$ be a sequence of continuous functions from a metric space (X, d) to a metric space (Y, ρ). Suppose that $\{f_k\}$ converges uniformly to $f \colon X \to Y$. Show that $f \colon X \to Y$ is continuous.

3. Which of the following sequences of functions converge uniformly on the interval $[0, 1]$?

(a) $f_n(x) = nx^2(1-x)^n$
(b) $f_n(x) = n^2x(1-x^2)^n$
(c) $f_n(x) = n^2x^3e^{-nx^2}$

4. Let $\{f_k\}$ be a sequence of linear maps $f_k\colon \mathbb{R}^n \to \mathbb{R}^m$ which are not identically zero, that is, for every $k \in \mathbb{Z}_{>0}$ there is $x = x_k$ such that $f_k(x) \neq 0$. Show that there is x (not depending on k) such that $f_k(x) \neq 0$ for all $k \in \mathbb{Z}_{>0}$.

5. Let $\{f_n\}$ be a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ having the property that $\{f_n(x)\}$ is unbounded for all $x \in \mathbb{Q}$. Prove that there is at least one $x \in \mathbb{Q}^c$ such that $\{f_n(x)\}$ is unbounded.

6. Let (X, d) be a complete metric space and let $(Y, \tilde{d})$ be a metric space. Let $\{f_n\}$ be a sequence of continuous functions from X to Y such that $\{f_n(x)\}$ converges for every $x \in X$. Prove that for every $\varepsilon > 0$ there exist $k \in \mathbb{Z}_{>0}$ and a non-empty open subset U of X such that $\tilde{d}(f_n(x), f_m(x)) < \varepsilon$ for all $x \in U$ and all $n, m \geq k$.

7. (a) Let (X, d) be a metric space and let (f_n) be a sequence of continuous functions, $f_n : X \to \mathbb{R}$, for $n \in \mathbb{Z}_{>0}$. Define what it means for the sequence f_n to converge uniformly to $f : X \to \mathbb{R}$.
   (a) (b)] Suppose that (f_n) is a sequence of continuous functions, $f_n : [0,1] \to \mathbb{R}$. Assume that (f_n) converges uniformly to $f : [0,1] \to \mathbb{R}$. Prove that $\int_0^x f_n(t)dt$ converges uniformly to $\int_0^x f(t)dt$, where $0 \leq x \leq 1$.
   (c) Let $f_n(x) = \dfrac{x^n}{1+x+x^n}$ for $x \in [0,1]$. Is the sequence (f_n) uniformly convergent on the interval $[0,1]$? Give a brief justification of your answer.

8. Let $X = C[0,1]$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$. Recall that the supremum metric on X is defined by
$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\}$$
and the L^1 metric on X is defined by
$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$
Consider the sequence $\{f_1, f_2, f_3, \ldots\}$ in X where $f_n(x) = nx^n(1-x)$ for $0 \leq x \leq 1$.
   (a) Determine whether $\{f_n\}$ converges in (X, d_1).
   (b) Determine whether $\{f_n\}$ converges in (X, d_∞).

   (You may use any standard results about limits of real sequences.)

9. (a) Let (X, d) be a metric space and let $\{f_n\}$ be a sequence of continuous functions, $f_n : X \to \mathbb{R}$, for $n \in \mathbb{Z}_{>0}$. Give the definition of uniform convergence of the sequence f_n to a function $f : X \to \mathbb{R}$.
   (b) Prove that if $\{f_n\}$ converges uniformly to $f : X \to \mathbb{R}$, then f is a continuous function.

(c) Let $f_n(x) = \dfrac{1-x^n}{1+x^n}$ for $x \in [0,1]$ and $n \in \mathbb{Z}_{>0}$. Find the pointwise limit f of the sequence $\{f_n\}$. Determine whether the sequence f_n is uniformly convergent to f or not on the interval $[0,1]$. Give brief reasons for your answer.

(d) Is the sequence (f_n) uniformly convergent on the interval $[0,1]$?

10. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $(f_1, f_2, \ldots)$ be a sequence of functions: $f_n \colon X \to Y$ for $n \in \mathbb{Z}_{>0}$.

    (a) Define what it means for the sequence $(f_1, f_2 \ldots)$ to converge uniformly to a function $f : X \to Y$.
    (b) Define what it means for a function $g \colon X \to Y$ to be bounded.
    (c) Prove that if each f_n is bounded and $(f_1, f_2, \ldots)$ converges uniformly to f, then f is also bounded.
    (d) Define $f_n : [0,1] \to \mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$ by

    $$f_n(x) = \frac{nx^2}{1+nx}, \qquad \text{for } x \in [0,1].$$

    Find the pointwise limit f of the sequence $(f_1, f_2, \ldots)$ and determine whether the sequence converges uniformly to f.

23.8.5 Compactness and completeness

1. Show that the completion of $(0,1)$ with the usual metric is $[0,1]$ with the usual metric.

2. Decide if the following metric spaces are complete:

    (a) $((0,\infty), d)$, where $d(x,y) = |x^2 - y^2|$ for $x, y \in (0,\infty)$.
    (b) $((-\pi/2, \pi/2), d)$, where $d(x,y) = |\tan x - \tan y|$ for $x, y \in (-\pi/2, \pi/2)$.

3. On $\mathbb{R}$ consider the metrics:

    $$d_1(x,y) = |\arctan x - \arctan y|,$$
    $$d_2(x,y) = \left|x^3 - y^3\right|.$$

    With which of these metrics is $\mathbb{R}$ complete? If $(\mathbb{R}, d_i)$ is not complete find its completion.

4. Which of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are compact? ($\mathbb{R}$ and $\mathbb{R}^2$ are considered with the usual metrics).

    (a) $A = \mathbb{Q} \cap [0,1]$
    (b) $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
    (c) $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}\}$
    (d) $D = \{(x,y) : |x| + |y| \leq 1\}$
    (e) $E = \{(x,y) : x \geq 1 \text{ and } 0 \leq y \leq 1/x\}$

5. Consider the following spaces:

(a) $\mathbb{R}$ with the metric $d_1(x,y) = \dfrac{|x-y|}{1+|x-y|}$;
(b) $\mathbb{R}$ with the metric $d_2(x,y) = |\arctan x - \arctan y|$;
(c) $\mathbb{R}$ with the metric $d_3(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$.

Is $(\mathbb{R}, d_i)$ compact?

6. Consider $C[0,1]$ with the usual d_∞ metric. Let

$$A = \{f \in C[0,1] \mid 0 = f(0) \le f(t) \le f(1) = 1 \text{ for all } t \in [0,1]\}.$$

Show that there is no finite $\frac{1}{2}$-net for A.

7. Let $X = (0,1]$ be equipped with the usual metric $d(x,y) = |x-y|$. Show that (X,d) is not complete. Let $\tilde{d}(x,y) = \|\dfrac{1}{x} - \dfrac{1}{y}\|$ for $x, y \in X$. Show that $\tilde{d}$ is a metric on X that is equivalent to d, and that $(X, \tilde{d})$ is complete.

8. Which of the following maps are contractions?

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$;
(b) $f : [0,\infty) \to [0,\infty)$, $f(x) = e^{-x}$;
(c) $f : [0,\infty) \to [0,\infty)$, $f(x) = e^{-e^x}$;
(d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$;
(e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(\cos x)$.

9. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \frac{1}{10}(8x + 8y, x + y),\ (x,y) \in \mathbb{R}^2.$$

Recall metrics $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $d_2((x_1,y_1),(x_2,y_2)) = [|x_1 - x_2|^2 + |y_1 - y_2|^2]^{1/2}$ and $d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Is f a contraction with respect to d_1? d_2? d_∞?

10. (a) Consider $X = (0,a]$ with the usual metric and $f(x) = x^2$ for $x \in X$. Find values of a for which f is a contraction and show that $f : X \to X$ does not have a fixed point.
(b) Consider $X = [1,\infty)$ with the usual metric and let $f(x) = x + \frac{1}{x}$ for $x \in X$. Show that $f : X \to X$ and $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$, but f does not have a fixed point. Reconcile (a) and (b) with Banach fixed point theorem.

11. (a) State the Banach fixed point theorem.
A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined as a *contraction* if there exists a constant c with $0 \le c < 1$ such that $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}$.
(b) (1) Use (a) to show that the equation $x + f(x) = a$ has a unique solution for each $a \in \mathbb{R}$.

(2) Deduce that $F:\mathbb{R}\to\mathbb{R}$ defined by $F(x)=x+f(x)$ is a bijection. (This should be easy).
(3) Show that F is continuous.
(4) Show that F^{-1} is continuous. (Hence F is a homeomorphism.)

12. Consider the map $f\colon \mathbb{R}^2\to\mathbb{R}^2$ given by

$$f(x,y)=\tfrac{1}{10}(8x+8y,x+y).$$

Recall metrics

$$\begin{aligned}
d_1((x_1,y_1),(x_2,y_2)&=|x_1-x_2|+|y_1-y_2|,\\
d_2((x_1,y_1),(x_2,y_2)&=(|x_1-x_2|^2+|y_1-y_2|^2)^{1/2},\\
d_\infty((x_1,y_1),(x_2,y_2)&=\max\{|x_1-x_2|,|y_1-y_2|\}
\end{aligned}$$

If f a contraction with respect to d_1? d_2? d_∞? Prove that your answers are correct.

13. Let $a\in\mathbb{R}_{>0}$. Let

$$f(x)=\frac{1}{2}\left(x+\frac{a}{x}\right),\qquad \text{for } x\in\mathbb{R}_{>0}.$$

(a) Show that if $x\in\mathbb{R}_{>0}$ then $f(x)\ge\sqrt{a}$. Hence f defines a function $f:X\to X$ where $X=[\sqrt{a},\infty)$.
(b) Show that f is a contraction mapping when X is given the usual metric.
(c) Fix $x_0>\sqrt{a}$ and $x_{n+1}=f(x_n)$, for $n\in\mathbb{Z}_{\ge0}$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

14. (a) State the Banach fixed point theorem.
(b) Verify that the mapping

$$f(x,y)=(\frac{x+y+1}{4},\frac{x-y+1}{4})$$

satisfies the conditions of the Banach fixed point theorem on the metric space $(\overline{B(0,1)},d)$, where d is the usual Euclidean metric and $\overline{B(0,1)}$ is the closed unit ball in $\mathbb{R}^2$ centred at the origin 0.
(c) Find directly the unique fixed point for f.

15. (a) State the Banach fixed point theorem.
(b) Let $X=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le1\}$.
Verify that the mapping $f\colon X\to X$ given by

$$f(x,y)=\big(\tfrac14(x+y+1),\tfrac14(x-y+1)\big)$$

satisfies the conditions of the Banach fixed point theorem.
(c) Find directly the unique fixed point for f.

16. (a) Show that there is exactly one continuous function $f : [0,1] \to \mathbb{R}$ which satisfies the equation

$$\big(f(x)\big)^3 - e^x\big(f(x)\big)^2 + \frac{f(x)}{2} = e^x.$$

(Hint: rewrite the equation as $f(x) = e^x + \frac{1}{2}\left(\frac{f(x)}{1+f(x)^2}\right)$.)

(b) Consider $C[0,a]$ with $a < 1$ and $T : C[0,a] \to C[0,a]$ given by

$$(Tf)(t) = \sin t + \int_0^t f(s)ds,\ t \in [0,a].$$

Show that T is a contraction. What is the fixed point of T?

(c) Find all $f \in C[0,\pi]$ which satisfy the equation

$$3f(t) = \int_0^t \sin(t-s)f(s)ds.$$

(d) Let $g \in C[0,1]$. Show that there exists exactly one $f \in C[0,1]$ which solves the equation

$$f(x) + \int_0^1 e^{x-y-1}f(y)dy = g(x), \quad \text{for all } x \in [0,1].$$

(Hint: Consider the metric $d(f,h) = \sup\{e^{-x}|f(x)-h(x)| \mid x \in [0,1]\}$.)

17. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. Let X be a complete metric space and let $f\colon X \to X$ be a α-contraction. Let $x \in X$, $x_0 = x$ and $x_{n+1} = f(x_n)$, for $n \in \mathbb{Z}_{\ge 0}$.

(a) Show that the sequence $x_0, x_1, x_2, \ldots$ converges in X.

Let $p = \lim_{n\to\infty} x_n$.

(b) Show that $d(x,p) \le \dfrac{d(x, f(x))}{1-\alpha}$.

(c) Show that $f(p) = p$.

18. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ be a continuous function which satisfies the Lipschitz condition with respect to the second variable: There exists $\alpha \in \mathbb{R}_{>0}$ such that

$$\text{if } (x,y_1),(x,y_2) \in U \quad \text{then} \quad |f(x,y_1) - f(x,y_2)| \le \alpha|y_1 - y_2|.$$

Show that if $(x_0,y_0) \in U$ then there exists $\delta \in \mathbb{R}_{>0}$ such that $y'(x) = f(x,y(x))$ has a unique solution $y\colon [x_0-\delta, x_0+\delta] \to \mathbb{R}$ such that $y(x_0) = y_0$.

19. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Let

$$\|f\| = \int |f| \qquad \text{and} \qquad d(f,g) = \|f-g\|,$$

for $f, g \in S$.

(a) Show that $\|\ \|\colon S \to \mathbb{R}_{\ge 0}$ is not a norm on S.

(b) Show that $d\colon S\times S\to \mathbb{R}_{\ge 0}$ is not a metric on S.

20. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k\to\mathbb{R}$. Let $\sum_{i\in\mathbb{Z}_{>0}} f_i$ be a series in S which is norm absolutely convergent. Show that there exists a full set in $\mathbb{R}^k$ on which $\sum_{i\in\mathbb{Z}_{>0}} f_i$ converges.

21. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k\to\mathbb{R}$. Let $\sum_{n\in\mathbb{Z}_{>0}} f_k$ be a series in S which is norm absolutely convergent. Show that $\sum_{n\in\mathbb{Z}_{>0}} f_n = 0$ almost everywhere if and only if the limit of the norms of the partial sums of f_n converge to 0.

22. Let L^1 be the set of functions which are equal almost everywhere to limits of norm absolutely convergent series in S, where S is the set of linear combinations of step functions $f\colon \mathbb{R}^k\to\mathbb{R}$. Define

$$\|f\| = \int f \qquad \text{and} \qquad d(f,g)=\|f-g\|, \quad \text{for } f,g\in L'.$$

(a) Show that $\|\ \|\colon L^1\to\mathbb{R}_{\ge 0}$ is a norm on L^1.
(b) Show that $d\colon L^1\times L^1\to\mathbb{R}_{\ge 0}$ is a metric on L^1.

23. Let I be a closed and bounded interval in $\mathbb{R}$. Let $x_1, x_2, x_3, \ldots$ be a sequence in I. Show that there exists a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of $x_1, x_2, x_3, \ldots$ such that $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ converges in I.

24. Let $X=\mathbb{R}$ with metric given by $d(x,y)=|x-y|$.

(a) Let $A=X$. Show that A is Cauchy compact but not bounded.
(b) Let $A=X$. Show that A is Cauchy compact but not cover compact.
(c) Let $A=X$. Show that A is Cauchy compact but not ball compact.
(d) Let $A=(0,1)\subseteq X$. Show that A is ball compact and not closed in X.
(e) Let $A=(0,1)\subseteq X$. Show that A is ball compact and not cover compact.
(f) Let $A=(0,1)\subseteq X$. Show that A is ball compact and not Cauchy compact.
(h) Let $A=(0,1)\subseteq X$ and let $B=A$. Show that B is closed in A but B is not Cauchy compact.
(g) Let $Y=\mathbb{R}$ with metric given by $\rho(x,y)=\min\{|x-y|,1\}$ and let $A=Y$. Show that A is bounded but not ball compact.

25. Let $(0,1)=\{x\in\mathbb{R}\mid 0<x<1\}$ with the standard metric. Show that X is not complete, is totally bounded and is not cover compact.

26. Let $C([0,1],\mathbb{R})=\{f\colon [0,1]\to\mathbb{R}\mid f \text{ is continuous}\}$ and let $d(f,g)=\sup\{|f(x)-g(x)|\mid x\in[0,1]\}$.

(a) Show that $d\colon C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ is a metric on $C([0,1,\mathbb{R})$.
(b) Let $A=\overline{B}_1(0)=\{f\in C([0,1],\mathbb{R})]\mid d(f,0)\le 1\}$. Show that A is closed and bounded.
(c) Show that A is not compact.

27. Let $K \subseteq \mathbb{R}$. Show that K is compact if and only if K is closed and bounded.

28. Let X be a compact metric space. Let $f\colon X \to \mathbb{R}$ be a continuous function. Show that f attains a maximum and a minimum value.

29. Let $f : X \to \mathbb{R}$. The function f is *upper semicontinuous* if f satisfies

$$\text{if } r \in \mathbb{R} \quad \text{then} \quad \{x \in X \mid f(x) < r\} \text{ is open.}$$

The function f is *lower semicontinuous* if f satisfies

$$\text{if} r \in \mathbb{R} \quad \text{then} \quad \{x \in X \mid f(x) > r\} \text{ is open.}$$

Assume that X is compact. Show that every upper semicontinuous function assumes a maximum value and every lower semicontinuous function assumes a minimum value.

30. Let $X = \mathbb{R}$ with metric given by $d(x, y) = \min\{|x - y|, 1\}$.
    (a) Show that X is bounded.
    (b) Show that X is not totally bounded.

31. Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let

$$f\colon [0, 2\pi) \to S^1 \qquad \text{be given by} \qquad f(x) = (\cos x, \sin x).$$

    (a) Show that f is continuous.
    (b) Show that f is a bijection.
    (c) Show that $f^{-1}\colon S^1 \to [0, 2\pi)$ is not continuous.

23.8.6 Connectedness

1. Let X_1 and X_2 be the subspaces of $\mathbb{R}$ given by $X_1 = \mathbb{R} - \{0\}$ and $X_2 = \mathbb{Q}$. Show that X_1 and X_2 are disconnected.

2. Let $A = (-\infty, 0)$ and $B = (0, \infty)$ as subsets of $\mathbb{R}$. Show that A is connected, B is connected and $A \cup B$ is not connected.

3. Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

4. Carefully state the Intermediate Value Theorem.

5. State and prove the Intermediate Value Theorem.

6. Let X be a connected topological space and let $f\colon X \to \mathbb{R}$ be a continuous function. Show that if $x, y \in X$ and $r \in \mathbb{R}$ such that $f(x) \le r \le f(y)$ then there exists $c \in X$ such that $f(c) = r$.

7. Let $X = \{(t, \sin(\pi t)) \mid t \in (0, 2]\}$.

   (a) Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\varphi(x, y) = x$. Show that $\varphi\colon X \to (0, 2]$ is a homeomorphism.
   (b) Show that X is connected.
   (c) Show that $\overline{X}$ is connected.
   (d) Show that $\overline{X}$ is not path connected.

   bigskip

8. Which of the following sets X are connected in $\mathbb{R}^2$?

   (a) Let $H = \{(x, y) \in \mathbb{R}^2 \mid xy = 1 \text{ and } x, y > 0\}$, $L = \{(x, 0) \mid x \in \mathbb{R}\}$, and $X = H \cup L$;

   (b) Let $C_n = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\}$ for $n \in \mathbb{Z}$, and $X = \bigcup_{n \in \mathbb{Z}_{>0}} C_n$.

9. Show that no two of the intervals (a, b), $(a, b]$, and $[a, b]$ are homeomorphic.

10. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic (where $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual topologies).

11. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 < 1\}$.
    Determine whether $X = A \cup B$, $Y = \overline{A} \cup \overline{B}$ and $Z = \overline{A} \cup B$ are connected subsets of $\mathbb{R}^2$ with the usual topology.

12. Let X be a connected topological space and let $f : X \to \mathbb{R}$ be a continuous function, where $\mathbb{R}$ has the usual topology. Show that if f takes only rational values, i.e. $f(X) \subseteq \mathbb{Q}$, then f is a constant function.

13. Show that $X = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is not homeomorphic to $\mathbb{R}$ (with the usual topologies). [Hint: consider the effect of removing points from X and $\mathbb{R}$.]

14. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 < 1\}$.
    Determine whether $X = A \cup B$, $Y = \overline{A} \cup \overline{B}$ and $Z = \overline{A} \cup B$ are connected subsets of $\mathbb{R}^2$ with the usual topology.

15. Let X be a connected topological space and let $f : X \to \mathbb{R}$ be a continuous function, where $\mathbb{R}$ has the usual topology. Show that if f takes only rational values, i.e. $f(X) \subseteq \mathbb{Q}$, then f is a constant function.

16. Explain why the following pairs of topological spaces are *not* homeomorphic. (Each has the topology induced from the usual embedding into a Euclidean space).

(a) $\mathbb{R}$ and S^1, where S^1 is the unit circle $\{(x,y) : x^2+y^2=1\}$.
(b) $(0,\infty)$ and $(0,1]$.
(c) $A = \{(0,y) : y \in \mathbb{R}\} \cup \{(x,0) : x \in \mathbb{R}\}$, $B = \{(0,y) : y \in \mathbb{R}, y \geq 0\} \cup \{(x,0) : x \in \mathbb{R}\}$.

17. Prove that no two of the following spaces are homeomorphic:

(i) $X = [-1,1]$ with the topology induced from $\mathbb{R}$;
(ii) $Y = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ with the topology induced from $\mathbb{R}^2$;
(iii) $Z = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ with the topology induced from $\mathbb{R}^2$.

18. Show that $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ is not homeomorphic to $\mathbb{R}$.

19. Let

$$A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\} \qquad \text{and}$$
$$B = \{(x,y) \in \mathbb{R}^2 : (x-2)^2+y^2 < 1\}.$$

Determine, with proof, whether $X = A \cup B$, $Y = \overline{A} \cup \overline{B}$ and $Z = \overline{A} \cup B$ are connected subsets of $\mathbb{R}^2$ with the usual topology.

20. Show that $\mathbb{Q}$, with the standard topology, is totally disconnected (i.e. each connected component contains only one point).

21. (a) Let (X, d_X) and (Y, d_Y) be a metric spaces and let $f\colon X \to Y$ be a function. Let $E \subseteq X$. Prove that if $f\colon X \to Y$ is continuous and E is connected then $f(E)$ is connected.
(b) Carefully state the intermediate value theorem.
(c) Prove the intermediate value theorem.