24 Problem list: New spaces from old

24.1 Subspaces

1. (restrictions of continuous functions are continuous) Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ with the subspace topology. Let $f: X \to Y$ be a continuous function. Show that
$$\begin{array}{rccc} g\colon & A & \to & Y \\ & a & \mapsto & f(a) \end{array} \quad \text{is continuous.}$$

2. (The subspace topology) Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. The *subspace topology on* Y is
$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$
Show that $\mathcal{T}_Y$ is a topology on Y.

3. (characterizing the subspace topology by continuity) Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ be a subset. Show that the subspace topology on Y is the minimal topology on Y such that the inclusion
$$\begin{array}{rccc} i\colon & Y & \longrightarrow & X \\ & y & \longmapsto & y \end{array} \quad \text{is continuous.}$$

4. (The subspace uniformity) Let $(X, \mathcal{X})$ be a uniform space and let Y be a subset of X. The *subspace uniformity on* Y is
$$\mathcal{X}_Y = \{V \cap (Y \times Y) \mid V \in \mathcal{X}\}.$$
Show that $\mathcal{X}_Y$ is a uniformity on Y.

5. (characterizing the subspace uniformity by uniform continuity) Let $(X, \mathcal{X})$ be a uniform space and let $Y \subseteq X$ be a subset. Show that the subspace uniformity on Y is the minimal uniformity on Y such that the inclusion
$$\begin{array}{rccc} i\colon & Y & \longrightarrow & X \\ & y & \longmapsto & y \end{array} \quad \text{is uniformly continuous.}$$

6. (A subspace of a vector space) Let X be a $\mathbb{K}$-vector space. A *subspace* of X is a subset $V \subseteq X$ such that

   (a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,
   (b) If $v \in V$ and $c \in \mathbb{K}$ then $cv \in V$.

   Show that V with the same operations of addition and scalar multiplication as in X is a vector space.

7. (A subspace of a normed vector space is a normed vector space) Let X be a normed vector space. Let $V \subseteq X$ be a subspace. Show that V is a normed vector space with the same norm.

8. (A subset of a metric space is a metric space) Let (X, d) be a metric space. Let $Y \subseteq X$ be a subset. Show that (Y, d) is a metric space.

24.2 Products

1. (characterizing the product topology by continuous functions) Let $(X, \mathcal{T})$ and $(Y, \mathcal{Q})$ be topological spaces. Show that the product topology on $X \times Y$ is the minimal topology such that the functions

$$\begin{array}{rccc} pr_1\colon & X\times Y & \to & X \\ & (x,y) & \mapsto & x \end{array} \quad\text{and}\quad \begin{array}{rccc} pr_2\colon & X\times Y & \to & Y \\ & (x,y) & \mapsto & y \end{array} \quad \text{are continuous.}$$

2. (characterizing the product uniformity by uniformly continuous functions) Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be uniform spaces. Show that the product uniformity on $X \times Y$ is the minimal uniformity such that the functions

$$\begin{array}{rccc} pr_1\colon & X\times Y & \to & X \\ & (x,y) & \mapsto & x \end{array} \quad\text{and}\quad \begin{array}{rccc} pr_2\colon & X\times Y & \to & Y \\ & (x,y) & \mapsto & y \end{array} \quad \text{are uniformly continuous.}$$

3. (products of continuous functions are continuous) Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be continuous functions. Show that the function $f_1 \times f_2$ given by

$$\begin{array}{rccc} f_1\times f_2\colon & X_1\times X_2 & \longrightarrow & Y_1\times Y_2 \\ & (x_1,x_2) & \longmapsto & (f_1(x_1), f_2(x_2)) \end{array} \quad \text{is continuous.}$$

4. (The product topology) Let $(X, \mathcal{T})$ and $(Y, \mathcal{Q})$ be topological spaces. Let $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. The *product topology on* $X \times Y$ is

$$\text{the topology } \mathcal{T}_{X\times Y} \text{ generated by} \qquad \{U \times V \mid U \in \mathcal{T}, V \in \mathcal{Q}\}.$$

Show that $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}, V \in \mathcal{Q}\}$ is a base of the topology $\mathcal{T}_{X\times Y}$.

5. (An open set in the product topology is not necessarily a product of open sets) Let $X = \mathbb{R}$ and $Y = \mathbb{R}$ so that $X \times Y = \mathbb{R}^2$. Let $Z = B_1((2, 2))$ be the ball of radius one centered at the point $(2, 2)$ in $\mathbb{R}^2$. show that there do not exist open sets U and V in $\mathbb{R}$ such that $Z = U \times V$.

6. (Metrics that produce the product topology) Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $Y = X_1 \times X_2$ and define

$$\begin{aligned} d((x_1,x_2),(y_1,y_2)) &= d_1(x_1,y_1) + d_2(x_2,y_2), \\ \rho((x_1,x_2),(y_1,y_2)) &= \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}, \\ \sigma((x_1,x_2),(y_1,y_2)) &= \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2} \end{aligned}$$

   (a) Show that (Y, d), (Y, ρ) and (Y, σ) are metric spaces.
   (b) Show that (Y, d), (Y, ρ) and (Y, σ) are the same as topological spaces.

7. (direct sums of vector spaces) Let X and Y be $\mathbb{K}$-vector spaces. The *direct sum* of X and Y is the $\mathbb{K}$-vector space $X \oplus Y$ given by the set $X \times Y$ with addition and scalar multiplication given by
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad \text{and} \qquad c(x, y) = (cx, cy),$$
for $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$ and $c \in \mathbb{K}$. Show that $X \oplus Y$ is a $\mathbb{K}$-vector space.

8. (Norms that produce the product topology) Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be normed vector spaces. Define functions $\| \cdot \|_1 \colon X \oplus Y \to \mathbb{R}_{\geq 0}$, $\| \cdot \|_2 \colon X \oplus Y \to \mathbb{R}_{\geq 0}$ and $\| \cdot \|_\infty \colon X \oplus Y \to \mathbb{R}_{\geq 0}$ by
$$\|(x, y)\|_1 = \|x\|_X + \|y\|_Y, \qquad \|(x, y)\|_2 = \sqrt{\|x\|_X^2 + \|y\|_Y^2}, \qquad \text{and}$$
$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}.$$
   (a) Show that $(X \oplus Y, \| \cdot \|_1)$, $(X \oplus Y, \| \cdot \|_2)$ and $(X \oplus Y, \| \cdot \|_\infty)$ are normed vector spaces. (See [Bre, Ch. 2 Ex. 2].)
   (b) Show that $(X \oplus Y, \| \cdot \|_1)$, $(X \oplus Y, \| \cdot \|_2)$ and $(X \oplus Y, \| \cdot \|_\infty)$ are the same as topological spaces. (See [Bre, Ch. 5 Ex. 2].)

24.3 The space $B(V, W)$ of bounded linear operators

1. ($B(V, W)$ is a normed vector space) Let V and W be normed vector spaces. Show that
$$B(V, W) = \{\text{linear transformations } T \colon V \to W \mid \|T\| < \infty\} \qquad \text{where}$$
$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0 \right\},$$
is a normed vector space.

2. (If W is complete then $B(V, W)$ is complete) Let V and W be normed vector spaces and let $B(V, W)$ be the vector space of bounded linear operators from V to W with norm given by
$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0 \right\}, \qquad \text{for } T \in B(V, W).$$
Show that if W is complete then $B(V, W)$ is complete.

3. (duals of normed vector spaces are complete) Let V with $\| \| \colon V \to \mathbb{R}_{\geq 0}$ be a normed vector space. Show that V^*, the dual of V, is complete.

4. (for linear operators, finite norm and uniformly continuous and continuous are all equivalent) Let V and W be normed vector spaces. Let $T \colon V \to W$ be a linear transformation from V to W. Show that the following are equivalent.
   (a) $\|T\| < \infty$.
   (b) $T \colon V \to W$ is uniformly continuous.
   (c) $T \colon V \to W$ is continuous.

5. (closed graph condition for continuity) Let X and Y be Banach spaces and let $\Lambda\colon X \to Y$ be a linear transformation.

$$\text{If}\quad \Gamma_\Lambda = \{(x, \Lambda(x)) \mid x \in X\} \text{ is closed in } X \times Y \qquad \text{then}\quad \Lambda \text{ is continuous.}$$

6. (limits and inverses of bounded linear operators) Let X and Y be Banach space.
   (a) Let $(\Lambda_1, \Lambda_2, \ldots)$ be a sequence of bounded linear operators from X to Y such that

   $$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} \Lambda_n(x) \text{ exists.} \qquad \text{Define} \quad \Lambda(x) = \lim_{n\to\infty} \Lambda_n(x).$$

   Show that $\quad \Lambda\colon X \to Y \quad$ is a bounded linear transformation.

   (b) If $\Lambda\colon X \to Y$ is a bijective bounded linear transformation then

   $$\Lambda^{-1}\colon Y \to X \quad \text{is a bounded linear transformation.}$$

7. (Baire category theorem, open dense version) Let (X, d) be a complete metric space. Show that if $U_1, U_2, U_3, \ldots$ are open dense subsets of X

$$\text{then} \qquad \bigcap_{n\in\mathbb{Z}_{>0}} U_n \text{ is dense in } X.$$

8. (uniform boundedness) Let X and Y be Banach spaces. Let $\mathcal{F} \subseteq B(X, Y)$. Then

$$\sup\{\|\Lambda\| \mid \Lambda \in \mathcal{F}\} < \infty \qquad \text{or} \quad \text{there exists a dense set } S \subseteq X$$

such that

$$\text{if } x \in S \quad \text{then} \quad \sup\{\|\Lambda(x)\| \mid \Lambda \in \mathcal{F}\} = \infty.$$

9. (open mapping) Let X and Y be Banach spaces. Let $\Lambda\colon X \to Y$ be a surjective bounded linear operator. Then Λ satisfies

$$\text{if } U \text{ is an open set in } X \quad \text{then} \quad \Lambda(U) \text{ is an open set in } Y.$$

10. (bounded on the unit ball implies uniformly bounded) Let X and Y be Banach spaces and let $\mathcal{F} \subseteq \mathcal{B}(X, Y)$. Show that if $\mathcal{F}$ satisfies

$$\text{if } x \in X \text{ and } \|x\| \le 1 \quad \text{then} \quad \sup\{\|\Lambda(x)\| \mid \Lambda \in \mathcal{F}\} < \infty$$

then

$$\sup\Big\{ \sup\{\|\Lambda(x)\| \mid \|x\| \le 1\} \;\Big|\; \Lambda \in \mathcal{F}\Big\} < \infty.$$

24.4 Function spaces and sequences of functions

1. (If Y is complete then bounded continuous functions from X to Y is complete) Let (X, d_X) and (Y, d_Y) be metric spaces and let

$$\mathcal{BC}(X,Y) = \{f\colon X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded in } Y\},$$

with $d_\infty\colon \mathcal{BC}(X,Y) \times \mathcal{BC}(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

   (a) Show that $\mathcal{BC}(X,Y)$ is a metric space.
   (b) Show that if Y is a complete metric space then $\mathcal{BC}(X,Y)$ is a complete metric space.

2. (bounded real valued functions is a complete metric space) Let (X,d) be a metric space and let

$$B(X) = \{f\colon X \to \mathbb{R} \mid f(X) \text{ is bounded}\},$$

with metric $d_\infty\colon B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

Show that $B(X)$ is a complete metric space.

3. (sequences of functions) Let (X,d) and (C,ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f,g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\geq 0}$.) Let

$$(f_1, f_2, \dots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq \ell}$ then $d(f_n(x), f(x)) < \epsilon$.

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $x \in X$ and $n \in \mathbb{Z}_{\geq \ell}$ then $d(f_n(x), f(x)) < \epsilon$.

   (a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

   (b) Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

4. (uniform convergence implies pointwise convergence) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\ge 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\ge 0}$.) Let

$$(f_1, f_2, \dots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \qquad \text{then} \qquad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Show that if $(f_1, f_2, \ldots)$ converges uniformly to f then $(f_1, f_2, \ldots)$ converges pointwise to f.

5. (pointwise convergence does not imply uniform convergence) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\}, \qquad (f_1, f_2, \dots) \text{ a sequence in } F$$

and let $f\colon X \to C$ be a function.

   (a) Show that if $(f_1, f_2, \dots)$ converges uniformly to f then $(f_1, f_2, \ldots)$ converges pointwise to f.

   (b) Let $X = C = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ with metric given by $d(x, y) = \rho(x, y) = |x - y|$. For $n \in \mathbb{Z}_{>0}$ let

$$f_n\colon \begin{array}{ccc} \mathbb{R}_{[0,1]} & \to & \mathbb{R}_{[0,1]} \\ x & \mapsto & x^n \end{array} \qquad \text{and let } f\colon \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$$

   be given by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

   Show that $(f_1, f_2, \ldots)$ converges pointwise to f but does not converge uniformly to f.

   GRAPH f_1, f_2, f_3, f_4 and f

6. (uniformly convergent sequences of continuous functions have continuous limits) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\ge 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\ge 0}$.) Let

$$(f_1, f_2, \dots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Show that if $f_1, f_2, \ldots$ are all continuous and $(f_1, f_2, \ldots)$ converges uniformly to f then f is continuous.

7. (the pointwise limit of continuous functions is not necessarily continuous) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\}, \qquad (f_1, f_2, \dots) \text{ a sequence in } F,$$

and let $f\colon X \to C$ be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

Show that if $f_1, f_2, \ldots$ are all continuous and $(f_1, f_2, \ldots)$ converges pointwise to f then f is not necessarily continuous.

24.5 Additional sample exam questions

24.5.1 Subspaces

1. Let X be a topological space and let $Y \subseteq X$ with the subspace topology. Show that

   (a) $B \subseteq Y$ is open in Y if and only if $B = Y \cap A$ for some set $A \subseteq X$ which is open in X.
   (b) $B \subseteq Y$ is closed in Y if and only if there exists $F \subseteq X$ closed in X such that $B = Y \cap F$.

2. Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Let $A \subseteq X$. Show that the restriction of f to A, $f|_A\colon A \to Y$ is continuous.

24.5.2 Products

1. Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces. Define the product metric d on $X_1 \times X_2 \times \cdots \times X_n$ and show that $(X_1 \times \cdots \times X_n, d)$ is a metric space.

2. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces. Show that a sequence $\overline{x_n} = (x_n^{(1)}, \ldots, x_n^{(\ell)})$ in $X_1 \times \cdots \times X_\ell$ converges if and only if each of the sequences $x_n^{(i)}$ (in X_i) converges.

3. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let $(X_1 \times \cdots \times X_\ell, d)$ be the product metric space. Let $\sigma\colon (X_1 \times \cdots \times X_\ell) \times (X_1 \times \cdots \times X_\ell) \to \mathbb{R}$ be given by

$$\sigma(x, y) = \max\{d_i(x_i, y_i) \mid 1 \le i \le \ell\}.$$

Show that σ is a metric on $X_1 \times \cdots \times X_\ell$ and d is equivalent to σ.

4. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let $(X_1 \times \cdots \times X_\ell, d)$ be the product metric space. Let $\rho\colon (X_1 \times \cdots \times X_\ell) \times (X_1 \times \cdots \times X_\ell) \to \mathbb{R}$ be given by

$$\rho(x, y) = \left(\sum_{i=1}^{\ell} d_i(x_i, y_i)^2\right)^{\frac{1}{2}}.$$

Show that ρ is a metric on $X_1 \times \cdots \times X_\ell$ and d is equivalent to ρ.

5. Let $X_1, \ldots, X_\ell$ be topological spaces and let $X_1 \times \cdots \times X_\ell$ have the product topology. Show that

   (a) If $A_1 \subseteq X_1, \ldots, A_\ell \subseteq X_\ell$ are open then $A_1 \times \cdots \times A_\ell \subseteq X_1 \times \cdots \times X_\ell$ is open.
   (b) If $F_1 \subseteq X_1, \ldots, F_\ell \subseteq X_\ell$ are closed then $F_1 \times \cdots \times F_\ell \subseteq X_1 \times \cdots X_\ell$ is closed.

6. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let d be the product metric on $X_1 \times \cdots \times X_\ell$. Show that the metric space topology on $(X_1 \times \cdots \times X_\ell, d)$ is the product topology for $X_1 \times \cdots \times X_\ell$, where $X_1, \ldots, X_\ell$ have the metric space topology.

7. Let (X, d), (Y_1, ρ_1) and (Y_2, ρ_2) be metric spaces. Let $f\colon X \to Y_1$ and $g\colon X \to Y_2$ be functions. Define $h\colon X \to Y_1 \times Y_2$ by $h(x) = (f(x), g(x))$. Let $a \in X$. Show that h is continuous if and only if f and g are continuous at a.

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ & & \big\downarrow f\times g \\ & & Y_1 \times Y_2 \end{array} \qquad \begin{array}{ccc} x & \mapsto & (x, x) \\ & & \downarrow \\ & & (f(x), g(x)) \end{array}$$

8. Let (X, d), (Y_1, ρ_1) and (Y_2, ρ_2) be metric spaces. Let $f\colon X \to Y_1$ and $g\colon X \to Y_2$ be functions. Define $h\colon X \to Y_1 \times Y_2$ by $h(x) = (f(x), g(x))$. Let $a \in X$. Show that h is continuous if and only if f and g are continuous.

9. Give an example of metric spaces X, Y and Z and a function $f\colon X \times Y \to Z$ such that

   (a) if $x \in X$ then $\begin{array}{rccc} \ell_x\colon & Y & \to & Z \\ & y & \mapsto & f(x,y) \end{array}$ is continuous,
   (b) if $y \in Y$ then $\begin{array}{rccc} r_y\colon & X & \to & Z \\ & x & \mapsto & f(x,y) \end{array}$ is continuous,
   (c) $f\colon X \times Y \to Z$ is not continuous.

10. Let (X_i, d_i) be a metric space for $1 \le i \le n$ and let $X = \prod_{i=1}^n X_i$. Define

$$d(x,y) = \left[\sum_{i=1}^n d_i(x_i,y_i)^2\right]^{1/2},$$
$$\overline{d}(x,y) = \max\{d_i(x_i,y_i) \mid 1 \le i \le n\},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in X$. Verify that d and $\overline{d}$ are metrics on X.

11. Let (X_n, d_n), $n \in \mathbb{Z}_{>0}$, be a sequence of metric spaces and let $X = \prod_{n\in\mathbb{Z}_{>0}} X_n$ be the cartesian product of the X_n's. (The elements of X are of the form $x = (x_1, x_2, \ldots)$ with $x_n \in X_n$.) For $x, y \in X$, define

$$d(x,y) = \sum_{n=1}^\infty \frac{1}{2^n}\left(\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}\right).$$

Show that (X,d) is a metric space.

12. Sketch the open ball $B(0,1)$ in the metric space $(\mathbb{R}^3, d_i)$, where d_i is defined by

$$d_1(x,y) = |x_1-y_1| + |x_2-y_2| + |x_3-y_3|$$
$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$
$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, |x_3-y_3|\}.$$

for $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3) \in \mathbb{R}^3$.

13. Let $X = \mathbb{R}^2$. For $x = (x_1,x_2)$ and $y = (y_1,y_2) \in X$ define

$$d_M(x,y) = \begin{cases} |x_2-y_2|, & \text{if } x_1 = y_1, \\ |x_1-y_1| + |x_2| + |y_2|, & \text{if } x_1 \ne y_1. \end{cases}$$

Also define

$$d_K(x,y) = \begin{cases} \|x-y\| & \text{if } x = ty \text{ for some } t \in \mathbb{R}; \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

where $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. (Can you give reasonable interpretations of the metrics d_M and d_K?)

Study the convergence of the sequence x_n in the spaces (X, d_M) and (X, d_K) if

(a) $x_n = (\frac{1}{n}, \frac{n}{n+1})$;
(b) $x_n = (\frac{n}{n+1}, \frac{n}{n+1})$;
(c) $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$.

14. Let (X, d_X) and (Y, d_Y) be metric spaces and A, B are dense subsets of X and Y, respectively. Show that $A \times B$ is dense in $X \times Y$.

15. Let (X, d_X) and (Y, d_Y) be metric spaces and let $(X \times Y, d)$ be the product metric space. Show that if $A \subseteq X$ and $B \subseteq Y$ are dense subsets then $A \times B$ is dense in $X \times Y$. Is it true that if U is dense in $X \times Y$ then $p(U)$ is dense in X and $p'(U)$ is dense in Y, where p, p' are the projections of $X \times Y$ to the two factors X, Y respectively? Prove this or give a counterexample.

16. Let (X, d_X) and (Y, d_Y) be metric spaces and let $(X \times Y, d)$ be the product metric space. Show that if $A \subseteq X$ and $B \subseteq Y$, then
$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

17. A topological space X is defined as *locally connected* if given any point $x \in X$ and an open set $V \subset X$ with $x \in V$ we can find a connected open set $U \subset V$ with $x \in U$.
    (a) Show that if X is locally connected, then all the connected components of X are open.
    (b) Show that any open subset $A \subset X$, where X is a vector space with a norm, is locally connected.

18. (a) Let (X, d) and (Y, d') be metric spaces. Show that $d^*((x, y), (u, v)) = d(x, u) + d'(y, v)$ defines a metric on $X \times Y$.
    (b) Prove that the map $f : (X, d) \to (X \times Y, d^*)$ given by $x \to (x, y_0)$ is an isometry from X to $f(X)$.
    (c) Use (b) to deduce that if (X, d) and (Y, d') are connected spaces, then $(X \times Y, d^*)$ is a connected space. (Hint: If $X \times Y = U \cup V$ with U, V disjoint, open, show that $f(X) \subset U$ or $f(X) \subset V$. Repeat for different points y_0.)

19. Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Show that
$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

20. Prove that if X and Y are path connected then $X \times Y$ is also path connected.

21. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $X \times Y$ have the product topology.
    (a) Show that if $E \subseteq X$ then $\overline{E^c} = (E^\circ)^c$ and $(E^c)^\circ = (\overline{E})^c$.
    (b) Let E be a open set in X. Show that E is a dense subset of X if and only if E^c is nowhere dense in X.
    (c) Let $U_1, U_2, \ldots$ be open dense subsets of X. Show that $\bigcup_{i \in \mathbb{Z}_{>0}} U_i$ is dense in X if and only if $\bigcap_{i \in \mathbb{Z}_{>0}} (U_i)^c$ has empty interior.
    (d) Show that an open set in $X \times Y$ cannot be expected to be of the form $A \times B$ with A open in X and B open in Y.
    (e) Show that if $A \subseteq X$ and $B \subseteq Y$ then
$$\overline{A} \times \overline{B} = \overline{A \times B} \qquad \text{and} \qquad A^\circ \times B^\circ = (A \times B)^\circ.$$

24.5.3 Function spaces

1. Let X be a nonempty set. Define the set of bounded functions $B(X, \mathbb{R})$ and the sup norm on $B(X, \mathbb{R})$. Show that $B(X, \mathbb{R})$, with this norm, is a normed vector space.

2. Let $a, b \in \mathbb{R}$ with $a < b$. Define the set of continuous functions $C([a, b], \mathbb{R})$ and the L^1-norm on $C([a, b], \mathbb{R})$. Show that $C([a, b], \mathbb{R})$, with this norm, is a normed vector space.

3. Let $a, b \in \mathbb{R}$ with $a < b$. Show that the set $C_{\mathrm{bd}}([a, b]), \mathbb{R})$ of bounded continuous functions is a metric subspace of $C([a, b], \mathbb{R})$ with the L^1-norm.

4. Let X be a topological space and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions.
   (a) Show that $f + g$ is continuous.
   (b) Show that $f \cdot g$ is continuous.
   (a) Show that $f - g$ is continuous.
   (d) Show that if g satisfies if $x \in X$ then $g(x) \neq 0$ then f/g is continuous.

5. Let (X, d) and (Y, d') be metric spaces and let $C_b(X, Y)$ be the set of bounded continuous functions $f: X \to Y$ with the metric $\rho: C_b(X, Y) \times C_b(X, Y) \to \mathbb{R}_{\geq 0}$ given by
$$\rho(f, g) = \sup\{d'(f(x), g(x)) \mid x \in X\}.$$
Show that $(C_b(X, Y), \rho)$ is a metric space.

6. Let S be the set of linear combinations of step functions $f: \mathbb{R}^k \to \mathbb{R}$. Let
$$\|f\| = \int |f| \qquad \text{and} \qquad d(f, g) = \|f - g\|,$$
for $f, g \in S$.
   (a) Show that $\|\ \|: S \to \mathbb{R}_{\geq 0}$ is not a norm on S.
   (b) Show that $d: S \times S \to \mathbb{R}_{\geq 0}$ is not a metric on S.

7. Let S be the set of linear combinations of step functions $f: \mathbb{R}^k \to \mathbb{R}$. Let $\sum_{i \in \mathbb{Z}_{>0}} f_i$ be a series in S which is norm absolutely convergent. Show that there exists a full set in $\mathbb{R}^k$ on which $\sum_{i \in \mathbb{Z}_{>0}} f_i$ converges.

8. Let S be the set of linear combinations of step functions $f: \mathbb{R}^k \to \mathbb{R}$. Let $\sum_{n \in \mathbb{Z}_{>0}} f_k$ be a series in S which is norm absolutely convergent. Show that $\sum_{n \in \mathbb{Z}_{>0}} f_n = 0$ almost everywhere if and only if the limit of the norms of the partial sums of f_n converge to 0.

9. Let L^1 be the set of functions which are equal almost everywhere to limits of norm absolutely convergent series in S, where S is the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Define
$$\|f\| = \int f \qquad \text{and} \qquad d(f,g) = \|f-g\|, \quad \text{for } f,g \in L^1.$$
   (a) Show that $\| \ \|\colon L^1 \to \mathbb{R}_{\ge 0}$ is a norm on L^1.
   (b) Show that $d\colon L^1 \times L^1 \to \mathbb{R}_{\ge 0}$ is a metric on L^1.

10. Let $X = C[0,1]$. Let $F\colon X \to \mathbb{R}$ be defined by $F(f) = f(0)$. Moreover, let $d_\infty(f,g) = \sup\{|f(x)-g(x)| \mid x \in [0,1]\}$ and $d_1(f,g) = \int_0^1 |f(x)-g(x)|dx$.
Is F continuous when X is equipped with (a) the metric d_∞, (b) the metric d_1?

11. Which of the following sequences of functions converge uniformly on the interval $[0,1]$?
   (a) $f_n(x) = nx^2(1-x)^n$
   (b) $f_n(x) = n^2x(1-x^2)^n$
   (c) $f_n(x) = n^2x^3e^{-nx^2}$

12. Let $\{f_k\}$ be a sequence of linear maps $f_k\colon \mathbb{R}^n \to \mathbb{R}^m$ which are not identically zero, that is, for every $k \in \mathbb{Z}_{>0}$ there is $x = x_k$ such that $f_k(x) \neq 0$. Show that there is x (not depending on k) such that $f_k(x) \neq 0$ for all $k \in \mathbb{Z}_{>0}$.

13. Let $\{f_n\}$ be a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ having the property that $\{f_n(x)\}$ is unbounded for all $x \in \mathbb{Q}$. Prove that there is at least one $x \in \mathbb{Q}^c$ such that $\{f_n(x)\}$ is unbounded.

14. Carefully define $B(V,W)$ and prove that if W is complete then $B(V,W)$ is complete.

15. (sequences of functions) Let (X,d) and (C,ρ) be metric spaces. Let
$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\ge 0} \cup \{\infty\} \quad \text{by}$$
$$d_\infty(f,g) = \sup\{\rho(f(x),g(x)) \mid x \in X\}.$$
(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\ge 0}$.) Let
$$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$
be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\ge N}$ then $d(f_n(x), f(x)) < \epsilon$.

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $x \in X$ and $n \in \mathbb{Z}_{\ge N}$ then $\rho(f_n(x), f(x)) < \epsilon$.

(a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} \rho(f_n(x), f(x)) = 0.$$

(b) Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

16. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $(f_1, f_2, \ldots)$ be a sequence of functions: $f_n \colon X \to Y$ for $n \in \mathbb{Z}_{>0}$.

 (a) Define what it means for the sequence $(f_1, f_2 \ldots)$ to converge uniformly to a function $f : X \to Y$.
 (b) Define what it means for a function $g \colon X \to Y$ to be bounded.
 (c) Prove that if each f_n is bounded and $(f_1, f_2, \ldots)$ converges uniformly to f, then f is also bounded.
 (d) Define $f_n : [0, 1] \to \mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$ by

$$f_n(x) = \frac{nx^2}{1 + nx}, \qquad \text{for } x \in [0, 1].$$

 Find the pointwise limit f of the sequence $(f_1, f_2, \ldots)$ and determine whether the sequence converges uniformly to f.

17. (a) Let (X, d) be a metric space and let $\{f_n\}$ be a sequence of continuous functions, $f_n : X \to \mathbb{R}$, for $n \in \mathbb{Z}_{>0}$. Give the definition of uniform convergence of the sequence f_n to a function $f : X \to \mathbb{R}$.
 (b) Prove that if $\{f_n\}$ converges uniformly to $f : X \to \mathbb{R}$, then f is a continuous function.
 (c) Let $f_n(x) = \dfrac{1 - x^n}{1 + x^n}$ for $x \in [0, 1]$ and $n \in \mathbb{Z}_{>0}$. Find the pointwise limit f of the sequence $\{f_n\}$. Determine whether the sequence f_n is uniformly convergent to f or not on the interval $[0, 1]$. Give brief reasons for your answer.
 (d) Is the sequence (f_n) uniformly convergent on the interval $[0, 1]$?

18. (a) Let (X, d) be a metric space and let (f_n) be a sequence of continuous functions, $f_n : X \to \mathbb{R}$, for $n \in \mathbb{Z}_{>0}$. Define what it means for the sequence f_n to converge uniformly to $f : X \to \mathbb{R}$.
 (a) (b)] Suppose that (f_n) is a sequence of continuous functions, $f_n : [0, 1] \to \mathbb{R}$. Assume that (f_n) converges uniformly to $f : [0, 1] \to \mathbb{R}$. Prove that $\int_0^x f_n(t)dt$ converges uniformly to $\int_0^x f(t)dt$, where $0 \le x \le 1$.
 (c) Let $f_n(x) = \dfrac{x^n}{1 + x + x^n}$ for $x \in [0, 1]$. Is the sequence (f_n) uniformly convergent on the interval $[0, 1]$? Give a brief justification of your answer.

19. Let $X = C[0, 1] = \{f \colon [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$. The supremum metric $d_\infty \colon X \times X \to \mathbb{R}_{\ge 0}$ and the L^1 metric $d_1 \colon X \times X \to \mathbb{R}_{\ge 0}$ are defined by

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} \qquad \text{and}$$
$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

 Consider the sequence $\{f_1, f_2, f_3, \ldots\}$ in X where $f_n(x) = nx^n(1 - x)$ for $0 \le x \le 1$.

(a) Determine whether $\{f_n\}$ converges in (X, d_1).
(b) Determine whether $\{f_n\}$ converges in (X, d_∞).

20. Determine whether the following sequences of functions converge uniformly.

(a) $f_n(x) = e^{-nx^2}, \quad x \in [0,1];$
(b) $g_n(x) = e^{-x^2/n}, \quad x \in [0,1].$
(c) $g_n(x) = e^{-x^2/n}, \quad x \in \mathbb{R}.$