26 Problem list: Vector spaces with topology

26.1 Norms and inner products

(1) (Parallelogram property) Let V be a vector space over a field $\mathbb{F}$ and let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a bilinear form. Let $\|\ \|^2\colon V\to\mathbb{F}$ be the quadratic form associated to $\langle,\rangle$. Prove carefully that

$$\text{if } x,y\in V \quad\text{then}\quad \|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$

(2) (Pythagorean theorem) Let V be a vector space over a field $\mathbb{F}$ and let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a bilinear form. Let $\|\ \|^2\colon V\to\mathbb{F}$ be the quadratic form associated to $\langle,\rangle$. Prove carefully that

$$\text{If } x,y\in V \text{ and } \langle x,y\rangle=0 \text{ and } \langle y,x\rangle=0 \quad\text{then}\quad \|x\|^2+\|y\|^2=\|x+y\|^2.$$

(3) (Reconstruction) Let V be a vector space over a field $\mathbb{F}$ and let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a symmetric bilinear form. Let $\|\ \|^2\colon V\to\mathbb{F}$ be the quadratic form associated to $\langle,\rangle$. Assume that $2\neq 0$ in $\mathbb{F}$. Prove carefully that

$$\text{If } x,y\in V \quad\text{then}\quad \langle x,y\rangle=\tfrac{1}{2}(\|x+y\|^2-\|x\|^2-\|y\|^2).$$

(4) Assume $n\in\mathbb{Z}_{>0}$ and V is a vector space over a field $\mathbb{F}$ with $\dim(V)=n$. Let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a bilinear form. Let $B=\{b_1,\ldots,b_n\}$ and $C=\{c_1,\ldots,c_n\}$ be bases of V and let P_{CB} be the change of basis matrix. Let G_B be the Gram matrix of $\langle,\rangle$ with respect to the basis B and let G_C be the Gram matrix of $\langle,\rangle$ with respect to C. Prove carefully that

$$G_C=P_{BC}^tG_BP_{CB},$$

(5) Let $\mathbb{F}$ be a field with an involutive automorphism $\overline{}\colon\mathbb{F}\to\mathbb{F}$ and let V be an $\mathbb{F}$-vector space. Prove carefully that a sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$ satisfies

(no isotropic vectors condition) If $v\in V$ and $\langle v,v\rangle=0$ then $v=0$.

if and only if it satsifies

(no isotropic subspaces condition) If W is a subspace of V then $W\cap W^\perp=0$.

(6) Let $\mathbb{F}$ be a field with an involutive automorphism $\overline{}\colon\mathbb{F}\to\mathbb{F}$. Let V be an $\mathbb{F}$-vector space with a sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$. Let $W\subseteq V$ be a subspace of V. Assume W is finite dimensional, that $(w_1,\ldots,w_k)$ is a basis of W and that G is the Gram matrix of $\langle,\rangle$ with respect to the basis $\{w_1,\ldots,w_k\}$. Prove carefully that the following are equivalent:

(a) A dual basis to $(w_1,\ldots,w_k)$ exists.
(b) G is invertible.
(c) $W\cap W^\perp=0$.
(d) The linear transformation

$$\begin{array}{rccc}\Psi_W\colon & W & \to & W^*\\ & v & \longmapsto & \varphi_v\end{array}\qquad\text{given by}\qquad \varphi_v(w)=\langle v,w\rangle,$$

is an isomorphism.

(7) Let $\mathbb{F}$ be a field with an involution $\overline{}\colon \mathbb{F} \to \mathbb{F}$ such that the fixed field $\mathbb{K} = \{a \in \mathbb{F} \mid a = \bar{a}\}$ is an ordered field.

$$\text{For } a \in \mathbb{F} \text{ define} \qquad |a|^2 = a\bar{a}.$$

Let V be an $\mathbb{F}$-vector space with a sesquilinear form such that

(a) If $x, y \in V$ then $\langle y, x\rangle = \overline{\langle x, y\rangle}$.
(b) If $x, \in V$ then $\langle x, x\rangle \in \mathbb{K}_{\geq 0}$.

Prove carefully that

(c) (Cauchy-Schwarz) If $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.
(d) (Triangle inequality) If $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$.

26.2 The Cauchy-Schwarz and triangle inequalities

1. (Cauchy-Schwarz and the triangle inequality) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

Show that

(a) If $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.
(b) If $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$.

2. (Pythagorean theorem) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

Show that

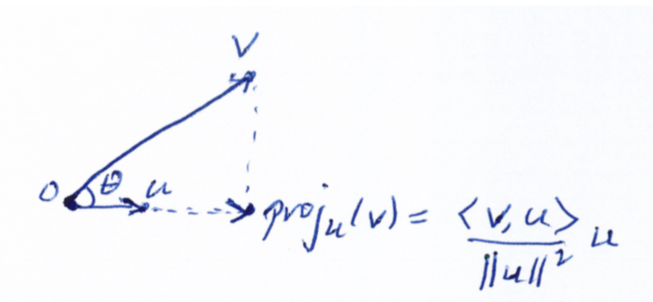
$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

3. (angles and projections) Let $(V, \langle,\rangle)$ be a inner product space and let $u, v \in V$.

The *angle between v and u* is $\theta \in [0, 2\pi)$ defined by

$$\cos(\theta) = \frac{\langle v, u\rangle}{\|v\|\,\|u\|} \qquad \text{and} \qquad \mathrm{proj}_u(v) = \left\langle v, \tfrac{u}{\|u\|}\right\rangle \frac{u}{\|u\|}.$$

is the *orthogonal projection of v onto u*.



(a) Use the Cauchy-Schwarz inequality to show that $0 \le \cos(\theta) < 1$ and show that $\|\mathrm{proj}_u(v)\| = \cos(\theta) \cdot \|v\|$.

(b) Let W be a finite dimensional subspace of V and let $\{u_1, \ldots, u_k\}$ be an orthonormal basis of W. The *orthogonal projection of v onto the subspace W* is

$$\mathrm{proj}_W(v) = \langle v, u_1\rangle u_1 + \cdots + \langle v, u_k\rangle u_k.$$

Show that $\mathrm{proj}_W(v)$ is independent of the choice of orthonormal basis.

4. Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

(a) (The Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

(b) (The triangle inequality) Show that if $x, y \in V$ then $\|x + y\| \le \|x\| + \|y\|$.

(c) (The Pythagorean theorem) Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

(d) (The parallelogram law) Show that

$$\text{if } x, y \in V \text{ then} \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(e) Show that if $(V, \|\ \|)$ is a normed vector space over $\mathbb{R}$ such that $\|\ \|: V \to \mathbb{R}_{\ge 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle,\rangle)$ with $\langle,\rangle: V \times V \to \mathbb{R}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1 + y\|^2 + \|x_2 + y\|^2 - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2.$$

To prove that $\langle cx, y\rangle = \lambda cx, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$.

(f) Show that if $(V, \|\ \|)$ is a normed vector space over $\mathbb{C}$ and $\|\ \|: V \to \mathbb{R}_{\ge 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle,\rangle)$ with $\langle,\rangle: V \times V \to \mathbb{C}$ given by

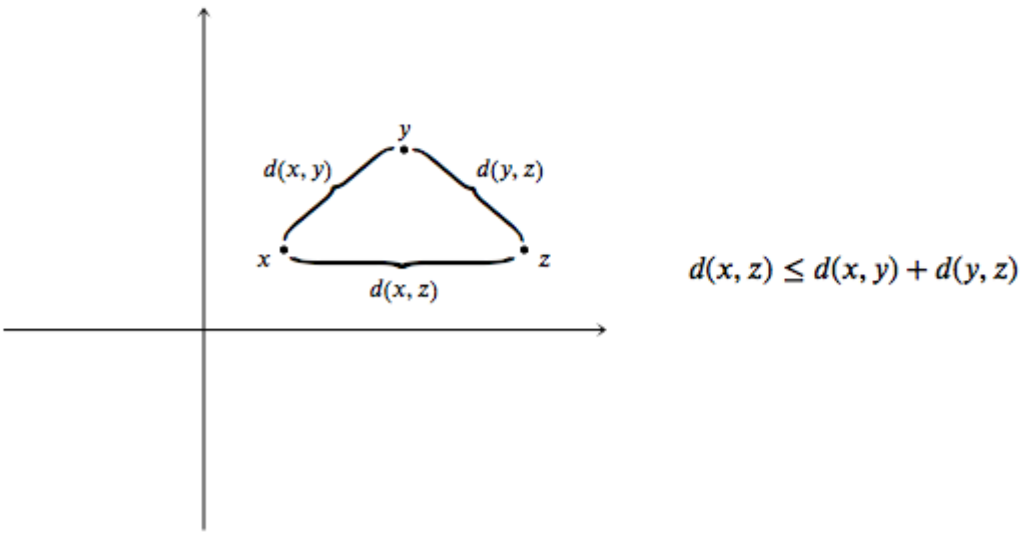
$$\langle x, y\rangle = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$.

26.3 Relating types of spaces

A *metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\ge 0}$ such that

(a) (diagonal condition) If $x \in X$ then $d(x,x) = 0$,
(b) (diagonal condition) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$,
(c) (symmetry condition) If $x, y \in X$ then $d(x,y) = d(y,x)$,
(d) (the triangle inequality) If $x, y, z \in X$ then $d(x,y) \le d(x,z) + d(z,y)$.



Distances between points in $\mathbb{R}^2$.

1. (positive definite inner product spaces are normed vector spaces) Let $(V, \langle , \rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

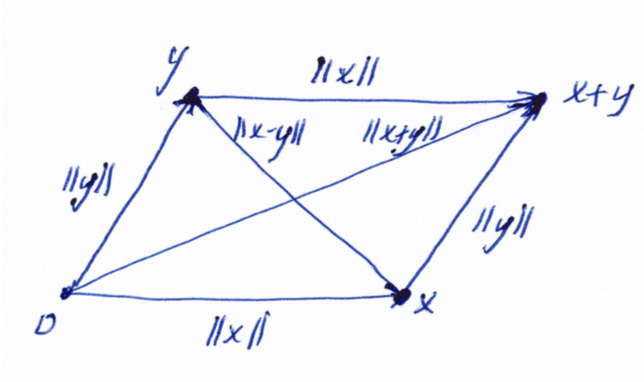
   Show that $(V, \| \, \|)$ is a normed vector space.

2. (inner product spaces from normed vector spaces: the parallelogram law)

   (a) Let $(V, \langle , \rangle)$ be a inner product space and let $\| \, \| \colon V \to \mathbb{R}_{\ge 0}$ be given by $\|v\|^2 = \langle v, v\rangle$. Show that

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

   (the sum of the squared lengths of the edges is the sum of the squared lengths of the daigonals).



(b) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{R}$ such that $\| \ \|\colon V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle , \rangle)$ with $\langle , \rangle\colon V \times V \to \mathbb{K}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\begin{aligned} \|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1 + y\|^2 + \|x_2 + y\|^2 \\ - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2. \end{aligned}$$

To prove that $\langle cx, y\rangle = c\langle x, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$. (See [Bre, Ch. 5 Ex. 3].)

(c) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{C}$ such that $\| \ \|\colon V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle , \rangle)$ with $\langle , \rangle\colon V \times V \to \mathbb{C}$ given by

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$. (See [Ru, Ch. 4 Ex. 11].)

3. (normed vector spaces are metric spaces) Let $(V, \| \ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d\colon V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

Show that (V, d) is a metric space.

4. Define the standard metric on $\mathbb{C}$ and show that $\mathbb{C}$, with this metric, is a metric space.

5. Let d be the standard metric on $\mathbb{C}$. Show that $\mathbb{R}$ is a metric subspace of $(\mathbb{C}, d)$.

6. Let X be a set. Define the standard metric on X and show that X, with this metric, is a metric space.

7. Let $(X, \| \ \|)$ be a normed vector space. Define the standard metric on X and show that X, with this metric, is a metric space.

8. Define the standard metric on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this metric, is a metric space.

9. Define the standard norm on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this norm, is a normed vector space.

10. Define the norm $\| \ \|_p$ on $\mathbb{R}^n$ and show that $(\mathbb{R}^n, \| \ \|_p)$ is a normed vector space.

11. Let X be a nonempty set. Define the set of bounded functions $B(X, \mathbb{R})$ and the sup norm on $B(X, \mathbb{R})$. Show that $B(X, \mathbb{R})$, with this norm, is a normed vector space.

12. Let $a, b \in \mathbb{R}$ with $a < b$. Define the set of continuous functions $C([a, b], \mathbb{R})$ and the L^1-norm on $C([a, b], \mathbb{R})$. Show that $C([a, b], \mathbb{R})$, with this norm, is a normed vector space.

13. Let $a, b \in \mathbb{R}$ with $a < b$. Show that the set $C_{\mathrm{bd}}([a, b]), \mathbb{R})$ of bounded continuous functions is a metric subspace of $C([a, b], \mathbb{R})$ with the L^1-norm.

14. (Polynomials of degree $\le n$ as a normed vector space) Fix a positive integer n. Denote by
$$\mathcal{P}_n = \{p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_1, \ldots, a_n \in \mathbb{R}\}.$$
For $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}_n$ set
$$\|p\| = \max\{|a_0|, |a_1|, \ldots, |a_n|\}.$$
Verify that $\| \ \|$ is a norm on $\mathcal{P}_n$.

15. Let (X, d) and (Y, d') be metric spaces and let $C_b(X, Y)$ be the set of bounded continuous functions $f \colon X \to Y$ with the metric $\rho \colon C_b(X, Y) \times C_b(X, Y) \to \mathbb{R}_{\ge 0}$ given by
$$\rho(f, g) = \sup\{d'(f(x), g(x)) \mid x \in X\}.$$
Show that $(C_b(X, Y), \rho)$ is a metric space.

16. Let S be the set of linear combinations of step functions $f \colon \mathbb{R}^k \to \mathbb{R}$. Let
$$\|f\| = \int |f| \qquad \text{and} \qquad d(f, g) = \|f - g\|,$$
for $f, g \in S$.

(a) Show that $\| \|\colon S \to \mathbb{R}_{\ge 0}$ is not a norm on S.
(b) Show that $d\colon S \times S \to \mathbb{R}_{\ge 0}$ is not a metric on S.

17. Let L^1 be the set of functions which are equal almost everywhere to limits of norm absolutely convergent series in S, where S is the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Define
$$\|f\| = \int f \qquad \text{and} \qquad d(f,g) = \|f-g\|, \quad \text{for } f,g \in L^1.$$
(a) Show that $\| \|\colon L^1 \to \mathbb{R}_{\ge 0}$ is a norm on L^1.
(b) Show that $d\colon L^1 \times L^1 \to \mathbb{R}_{\ge 0}$ is a metric on L^1.

18. (metric spaces are not always separable) A metric space (or topological space) is *separable* if it has a countable dense set.
(a) Show that $\mathbb{R}$ with the standard topology is separable.
(b) Show that $\mathbb{R}$ with the discrete topology is not separable.
(b) Show that $\mathbb{R}^n$ is separable.
(c) Show that ℓ^1 is separable.
(d) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is separable.
(e) Show that ℓ^∞ is not separable.

26.4 Triangle inequalities

1. (Cauchy-Schwarz and triangle inequalities in $\mathbb{R}^n$) Let $x, y \in \mathbb{R}^n$. Prove the following:
(a) (Lagrange's identity) $|x|^2 \cdot |y|^2 - \langle x,y\rangle^2 = \frac{1}{2}\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2$.
(b) (Cauchy-Schwarz inequality) $\langle x,y\rangle \le |x| \cdot |y|$.
(c) (triangle inequality) $|x+y| \le |x| + |y|$.

2. (Cauchy-Schwarz and triangle inequalities in inner product spaces) Let $(V, \langle, \rangle)$ be a positive definite inner product space.
(a) (Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x,y\rangle| \le \|x\| \cdot \|y\|$.
(b) (triangle inequality) Showthat if $x, y \in V$ then $\|x+y\| \le \|x\| + \|y\|$.

3. (Hölder and Minkowski inequalities) Let $q \in \mathbb{R}_{\ge 1}$ and let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$.
(a) (Young's inequality) Show that if $a, b \in \mathbb{R}_{>0}$ then $a^{\frac{1}{p}}\, b^{\frac{1}{q}} \le \frac{1}{p}\, a + \frac{1}{q}\, b$.
(b) (Hölder inequality for $\mathbb{R}^n$) Show that if $x, y \in \mathbb{R}^n$ then $|\langle x,y\rangle| \le \|x\|_p \|y\|_q$.
(c) (Minkowski inequality for $\mathbb{R}^n$) Show that if $x, y \in \mathbb{R}^n$ then $\|x+y\|_p \le \|x\|_p + \|y\|_p$.
(d) (Hölder inequality) Show that if $x \in \ell^p$ and $y \in \ell^q$ then $|\langle x,y\rangle| \le \|x\|_p \|y\|_q$.
(e) (Minkowski inequality) Show that if $x \in \ell^p$ and $y \in \ell^q$ then $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

26.5 Subspaces and products

1. (A subspace of a vector space) Let X be a $\mathbb{K}$-vector space. A *subspace* of X is a subset $V \subseteq X$ such that

   (a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,

   (b) If $v \in V$ and $c \in \mathbb{K}$ then $cv \in V$.

   Show that V with the same operations of addition and scalar multiplication as in X is a vector space.

2. (A subspace of a normed vector space is a normed vector space) Let X be a normed vector space. Let $V \subseteq X$ be a subspace. Show that V is a normed vector space with the same norm.

3. (A subset of a metric space is a metric space) Let (X, d) be a metric space. Let $Y \subseteq X$ be a subset. Show that (Y, d) is a metric space.

4. (direct sums of vector spaces) Let X and Y be $\mathbb{K}$-vector spaces. The *direct sum* of X and Y is the $\mathbb{K}$-vector space $X \oplus Y$ given by the set $X \times Y$ with addition and scalar multiplication given by
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad \text{and} \qquad c(x, y) = (cx, cy),$$
   for $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$ and $c \in \mathbb{K}$. Show that $X \oplus Y$ is a $\mathbb{K}$-vector space.

5. (Norms that produce the product topology) Let $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ be normed vector spaces. Define functions $\|\cdot\|_1 \colon X \oplus Y \to \mathbb{R}_{\geq 0}$, $\|\cdot\|_2 \colon X \oplus Y \to \mathbb{R}_{\geq 0}$ and $\|\cdot\|_\infty \colon X \oplus Y \to \mathbb{R}_{\geq 0}$ by
$$\|(x, y)\|_1 = \|x\|_X + \|y\|_Y, \qquad \|(x, y)\|_2 = \sqrt{\|x\|_X^2 + \|y\|_Y^2}, \qquad \text{and}$$
$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}.$$

   (a) Show that $(X \oplus Y, \|\cdot\|_1)$, $(X \oplus Y, \|\cdot\|_2)$ and $(X \oplus Y, \|\cdot\|_\infty)$ are normed vector spaces.

   (b) Show that $(X \oplus Y, \|\cdot\|_1)$, $(X \oplus Y, \|\cdot\|_2)$ and $(X \oplus Y, \|\cdot\|_\infty)$ are the same as topological spaces.

26.6 The space $B(V, W)$ of bounded linear operators

1. ($B(V, W)$ is a normed vector space) Let V and W be normed vector spaces. Show that
$$B(V, W) = \{\text{linear transformations } T \colon V \to W \mid \|T\| < \infty\} \qquad \text{where}$$
$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0\right\},$$
   is a normed vector space.

2. (If W is complete then $B(V,W)$ is complete) Let V and W be normed vector spaces and let $B(V,W)$ be the vector space of bounded linear operators from V to W with norm given by

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0\right\}, \qquad \text{for } T \in B(V,W).$$

Show that if W is complete then $B(V,W)$ is complete.

3. (duals of normed vector spaces are complete) Let V with $\| \ \|: V \to \mathbb{R}_{\geq 0}$ be a normed vector space. Show that V^*, the dual of V, is complete.

4. (If Y is complete then bounded continuous functions from X to Y is complete) Let (X, d_X) and (Y, d_Y) be metric spaces and let

$$\mathcal{BC}(X,Y) = \{f: X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded in } Y\},$$

with $d_\infty: \mathcal{BC}(X,Y) \times \mathcal{BC}(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

(a) Show that $\mathcal{BC}(X,Y)$ is a metric space.
(b) Show that if Y is a complete metric space then $\mathcal{BC}(X,Y)$ is a complete metric space.

5. (bounded real valued functions is a complete metric space) Let (X,d) be a metric space and let

$$B(X) = \{f: X \to \mathbb{R} \mid f(X) \text{ is bounded}\},$$

with metric $d_\infty: B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

Show that $B(X)$ is a complete metric space.

6. (for linear operators, finite norm and uniformly continuous and continuous are all equivalent) Let V and W be normed vector spaces. Let $T: V \to W$ be a linear transformation from V to W. Show that the following are equivalent.

(a) $\|T\| < \infty$.
(b) $T: V \to W$ is uniformly continuous.
(c) $T: V \to W$ is continuous.

7. (closed graph condition for continuity) Let X and Y be Banach spaces and let $\Lambda: X \to Y$ be a linear transformation.

$$\text{If} \quad \Gamma_\Lambda = \{(x, \Lambda(x)) \mid x \in X\} \text{ is closed in } X \times Y \qquad \text{then} \quad \Lambda \text{ is continuous.}$$

8. (limits and inverses of bounded linear operators) Let X and Y be Banach space.

   (a) Let $(\Lambda_1, \Lambda_2, \ldots)$ be a sequence of bounded linear operators from X to Y such that

   $$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} \Lambda_n(x) \text{ exists.} \qquad \text{Define} \quad \Lambda(x) = \lim_{n\to\infty} \Lambda_n(x).$$

   Show that $\quad \Lambda\colon X \to Y \quad$ is a bounded linear transformation.

   (b) If $\Lambda\colon X \to Y$ is a bijective bounded linear transformation then

   $$\Lambda^{-1}\colon Y \to X \quad \text{is a bounded linear transformation.}$$

9. (Baire category theorem, open dense version) Let (X, d) be a complete metric space. Show that if $U_1, U_2, U_3, \ldots$ are open dense subsets of X

   $$\text{then} \qquad \bigcap_{n\in\mathbb{Z}_{>0}} U_n \text{ is dense in } X.$$

10. (uniform boundedness) Let X and Y be Banach spaces. Let $\mathcal{F} \subseteq B(X, Y)$. Then

    $$\sup\{\|\Lambda\| \mid \Lambda \in \mathcal{F}\} < \infty \qquad \text{or} \quad \text{there exists a dense set } S \subseteq X$$

    such that

    $$\text{if } x \in S \quad \text{then} \quad \sup\{\|\Lambda(x)\| \mid \Lambda \in \mathcal{F}\} = \infty.$$

11. (open mapping) Let X and Y be Banach spaces. Let $\Lambda\colon X \to Y$ be a surjective bounded linear operator. Then Λ satisfies

    $$\text{if } U \text{ is an open set in } X \quad \text{then} \quad \Lambda(U) \text{ is an open set in } Y.$$

12. (bounded on the unit ball implies uniformly bounded) Let X and Y be Banach spaces and let $\mathcal{F} \subseteq \mathcal{B}(X, Y)$. Show that if $\mathcal{F}$ satisfies

    $$\text{if } x \in X \text{ and } \|x\| \leq 1 \quad \text{then} \quad \sup\{\|\Lambda(x)\| \mid \Lambda \in \mathcal{F}\} < \infty$$

    then

    $$\sup\left\{ \sup\{\|\Lambda(x)\| \mid \|x\| \leq 1\} \;\middle|\; \Lambda \in \mathcal{F}\right\} < \infty.$$

26.7 The spaces ℓ^p

1. Let $p \in \mathbb{R}_{>1}$ and define $q \in \mathbb{R}_{>1}$ by $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) Define the normed vector space ℓ^p.
   (b) Show that ℓ^p is a Banach space.
   (c) Prove that the dual of ℓ^p is ℓ^q.

2. Let $p \in \mathbb{R}_{\geq 1}$ and define

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_p = \left(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\right)^{1/p}$$

   for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$.
   (a) Show that if $p \leq q$ then $\ell^p \subseteq \ell^q$.
   (b) Show that if $p \neq q$ then $\ell^p \neq \ell^q$.

3. Let $p \in \mathbb{R}_{>1}$. Let $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis of ℓ^p.

4. (Containment of ℓ^p-spaces) Let $p, s \in \mathbb{R}_{>1}$. Show that if $p \leq s$ then $\ell^p \subseteq \ell^s$.

5. (ℓ^p-spaces depend on p) Let $p, s \in \mathbb{R}_{>1}$. Show that if $p \neq s$ then $\ell^p \neq \ell^s$.

6. (the dual of $\mathbb{R}^2$ in the $\| \ \|_p$ norm) Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional, say $\phi(x_1, x_2) = ax_2 + bx_2$. Give a direct proof that
   (a) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_1 = |x_1| + |x_2|$ then the corresponding operator norm is $\|\phi\|_\infty = \max\{|a|, |b|\}$.
   (b) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ then the corresponding operator norm is $\|\phi\|_1 = |a| + |b|$.
   (b) If $p \in \mathbb{R}_{>1}$ and $\mathbb{R}^2$ is endowed with the norm $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ then the corresponding operator norm is $\|\phi\|_p = (|a|^q + |b|^q)^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

7. (Dual of an ℓ^p-space) Let $p \in \mathbb{R}_{>1}$. Show that $(\ell^p)^* = \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

8. (Dual of c_0) Show that $(c_0)^* = \ell^1$.

9. (Dual of ℓ^1) Show that $(\ell^1)^* = \ell^\infty$.

10. (Dual of ℓ^∞) Show that $(\ell^\infty)^* \neq \ell^1$.

11. (ℓ^p is complete) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is a complete metric space.

12. (ℓ^1 is complete) Show that ℓ^1 is a complete metric space.

13. (ℓ^∞ is complete) Show that ℓ^∞ is a complete metric space.

14. (c_0 is complete) Show that c_0 is the completion of

$$c_c = \{(x_1, x_2, \ldots) \in \ell^\infty \quad \text{all but a finite number of } x_i \text{ are } 0\}$$

the space of sequences that are eventually 0. (IS THE esssup NORM AND THE SUP NORM THE SAME FOR COUNTING MEASURE? SEE Theorem 1.3 on the page http://www.ms.unimelb.edu.a

15. (The completion of c_c with respect to $\| \|_p$) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is the completion of

$$c_c = \{(x_1, x_2, \ldots) \in \ell^p \quad \text{all but a finite number of } x_i \text{ are } 0\}$$

the space of sequences that are eventually 0.

16. (the closure of the span of the standard basis in ℓ^p) Let

$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots,$$

and

$$\text{let } p \in \mathbb{R}_{>1}. \qquad \text{Show that,} \quad \text{in } \ell^p, \qquad \overline{\mathrm{span}\{e_1, e_2, \ldots\}} = \ell^p.$$

17. (the closure of the span of the standard basis in ℓ^1) Let

$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots.$$

Show that,

$$\text{in } \ell^1, \qquad \overline{\mathrm{span}\{e_1, e_2, \ldots\}} = \ell^1.$$

18. (the closure of the span of the standard basis in ℓ^∞) Let

$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots.$$

Show that,

$$\text{in } \ell^\infty, \qquad \overline{\mathrm{span}\{e_1, e_2, \ldots\}} = c_0.$$

19. (weak convergence of of the standard basis in ℓ^p) Let

$$e_1 = (1,0,0,0,0,\ldots), \quad e_2 = (0,1,0,0,0,\ldots), \quad e_3 = (0,0,1,0,0,\ldots), \quad \ldots,$$

and let $p \in \mathbb{R}_{>1}$. Show that

$$\text{in } \ell^p, \quad \text{the sequence } (e_1, e_2, e_3, \ldots) \text{ weakly converges weakly to } 0.$$

20. (weak convergence of of the standard basis in ℓ^p) Let

$$e_1 = (1,0,0,0,0,\ldots), \quad e_2 = (0,1,0,0,0,\ldots), \quad e_3 = (0,0,1,0,0,\ldots), \quad \ldots,$$

Show that,

$$\text{in } \ell^1, \ (e_1, e_2, e_3, \ldots) \text{ does not have any weakly convergent subsequence.}$$

26.8 Bilinear forms

1. (complex positive definite bilinear forms are skew symmetric) Let V be a $\mathbb{C}$-vector space. Let $f\colon V \times V \to \mathbb{C}$ be a function such that

   (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1 + c_2v_2, w) = c_1 f(v_1, w) + c_2 f(v_2, w)$.
   (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1 + c_2w_2) = c_1 f(v, w_1) + c_2 f(v, w_2)$.
   (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geq 0}$.

   Show that

   (A) If $v \in V$ then $f(v, v) = -f(v, v)$,
   (B) If $v \in V$ then $f(v, v) = 0$,
   (C) If $v, w \in V$ then $f(v, w) = -f(w, v)$.

2. (nondegenerate complex positive definite bilinear forms) Let $V = \mathbb{C}\text{-span}\{e_1, e_2\}$ so that $V \cong \mathbb{C}^2$. Show that there exists $f\colon V \times V \to \mathbb{C}$ such that

   (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1 + c_2v_2, w) = c_1 f(v_1, w) + c_2 f(v_2, w)$.
   (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1 + c_2w_2) = c_1 f(v, w_1) + c_2 f(v, w_2)$.
   (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geq 0}$,
   (d) If $v \in V$ and $v \neq 0$ then there exists $w \in V$ such that $f(v, w) \neq 0$.

3. (complex symmetric positive definite bilinear forms are zero) Let V be a $\mathbb{C}$-vector space. Let $f\colon V \times V \to \mathbb{C}$ be a function such that

   (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1 + c_2v_2, w) = c_1 f(v_1, w) + c_2 f(v_2, w)$.
   (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1 + c_2w_2) = c_1 f(v, w_1) + c_2 f(v, w_2)$.
   (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geq 0}$.

(d) If $v, w \in V$ then $f(v, w) = f(w, v)$.

Show that if $v, w \in V$ then $f(v, w) = 0$.

4. (nondegenerate real positive definite symmetric bilinear forms) Give an example of a nonzero $\mathbb{R}$-vector space and a function $f\colon V \times V \to \mathbb{R}$ such that

   (a) If $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2, w \in V$ then $f(c_1v_1 + c_2v_2, w) = c_1f(v_1, w) + c_2f(v_2, w)$.
   (b) If $c_1, c_2 \in \mathbb{R}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1 + c_2w_2) = c_1f(v, w_1) + c_2f(v, w_2)$.
   (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geq 0}$.
   (d) If $v, w \in V$ then $f(v, w) = f(w, v)$.
   (e) If $v \in V$ and $v \neq 0$ then there exists $w \in V$ such that $f(v, w) \neq 0$.

26.9 Cauchy-Schwarz and triangle inequalities

1. (Cauchy-Schwarz, the triangle inequality and the Pythagorean theorem) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

   Show that

   (a) If $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.
   (b) If $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$.

2. (Pythagorean theorem) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

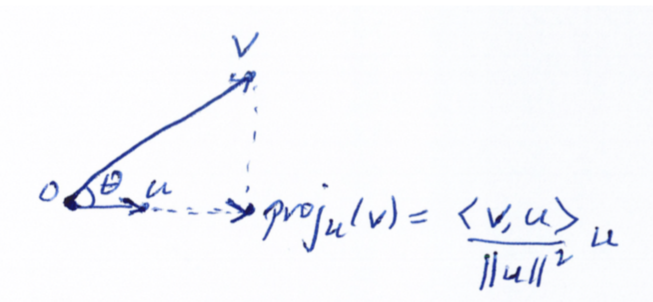
   Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

3. (angles and projections) Let $(V, \langle,\rangle)$ be a inner product space and let $u, v \in V$.
   The *angle between v and u* is $\theta \in [0, 2\pi)$ defined by

$$\cos(\theta) = \frac{\langle v, u\rangle}{\|v\|\,\|u\|} \qquad \text{and} \qquad \mathrm{proj}_u(v) = \left\langle v, \tfrac{u}{\|u\|}\right\rangle \frac{u}{\|u\|}.$$

   is the *orthogonal projection of v onto u*.



(a) Use the Cauchy-Schwarz inequality to show that $0 \le \cos(\theta) < 1$ and show that $\|\mathrm{proj}_u(v)\| = \cos(\theta)\cdot\|v\|$.

(b) Let W be a finite dimensional subspace of V and let $\{u_1, \ldots, u_k\}$ be an orthonormal basis of W. The *orthogonal projection of v onto the subspace W* is

$$\mathrm{proj}_W(v) = \langle v, u_1\rangle u_1 + \cdots + \langle v, u_k\rangle u_k.$$

Show that $\mathrm{proj}_W(v)$ is independent of the choice of orthonormal basis.

4. (positive definite inner product spaces are normed vector spaces) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

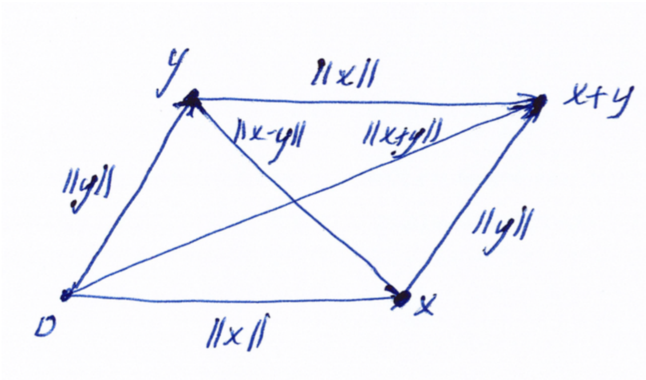
Show that $(V, \|\ \|)$ is a normed vector space.

5. (inner product spaces from normed vector spaces: the parallelogram law)

(a) Let $(V, \langle,\rangle)$ be a inner product space and let $\|\ \|\colon V \to \mathbb{R}_{\ge 0}$ be given by $\|v\|^2 = \langle v, v\rangle$. Show that

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(the sum of the squared lengths of the edges is the sum of the squared lengths of the daigonals).



(b) Show that if $(V, \|\ \|)$ is a normed vector space over $\mathbb{R}$ such that $\|\ \|\colon V \to \mathbb{R}_{\ge 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle,\rangle)$ with $\langle,\rangle\colon V \times V \to \mathbb{K}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\begin{aligned}\|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1 + y\|^2 + \|x_2 + y\|^2 \\ - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2.\end{aligned}$$

To prove that $\langle cx, y\rangle = c\langle x, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$. (See [Bre, Ch. 5 Ex. 3].)

(c) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{C}$ such that $\| \ \| \colon V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle , \rangle)$ with $\langle , \rangle \colon V \times V \to \mathbb{C}$ given by

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$. (See [Ru, Ch. 4 Ex. 11].)

6. Let $(V, \langle , \rangle)$ be an inner product space and let $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in V$. Show that if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

7. Let $(V, \| \ \|)$ be a normed vector space. Define

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

for $x, y \in V$. Show that if $\| \ \|$ satisfies if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ then $\langle , \rangle$ is an inner product on V.

8. (inner product spaces and the parallelogram law) Let $(V, \langle , \rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

(a) (The Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

(b) (The triangle inequality) Show that if $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$.

(c) (The Pythagorean theorem) Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x+y\|^2.$$

(d) (The parallelogram law) Show that

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(e) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{R}$ such that $\| \ \| \colon V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle , \rangle)$ with $\langle , \rangle \colon V \times V \to \mathbb{R}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1 + y\|^2 + \|x_2 + y\|^2 - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2.$$

To prove that $\langle cx, y\rangle = \lambda cx, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$.

(f) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{C}$ and $\| \ \| \colon V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle, \rangle)$ with $\langle, \rangle \colon V \times V \to \mathbb{C}$ given by

$$\langle x, y \rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v \rangle$.

9. (normed vector spaces are metric spaces) Let $(V, \| \ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d \colon V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x,y) = \|x-y\|.$$

Show that (V, d) is a metric space.

10. Let $(V, \langle, \rangle)$ be an inner product space. Show that if $x \in V$ and $x \neq 0$ then

$$\|x\| = \sup\left\{ \frac{|\langle x, y \rangle|}{\|y\|} \ \Big| \ y \in V \text{ and } y \neq 0 \right\}.$$

26.10 Examples of Banach spaces and Hilbert spaces

1. ($\mathbb{R}^n$ as a Hilbert space) Let A be a $n \times m$ real symmetric matrix with positive eigenvalues. Show that $\mathbb{R}^n$ with $\langle, \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle X, Y \rangle = X^T A Y, \qquad \text{is a real inner product space.}$$

2. ($\mathbb{C}^n$ as a Hilbert space) Let B be a $n \times n$ Hermitian matrix with positive eigenvalues. Show that $\mathbb{C}^n$ with $\langle, \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle X, Y \rangle = X^T B Y, \qquad \text{is a complex inner product space.}$$

3. (Norms on finite dimensional spaces are equivalent) Let V be a finite dimensional vector space. Let $\| \ \|_1$ and $\| \ \|_2$ be norms on V. Show that if $y \in V$ and $(x_1, x_2, \ldots)$ is a sequence in V then

$$\lim_{n\to\infty} \|x_n - y\|_1 = 0 \qquad \text{if and only if} \qquad \lim_{n\to\infty} \|x_n - y\|_2 = 0.$$

4. (The space $(\mathbb{R}^n, \| \ \|_p)$) Let $p \in \mathbb{R}_{\geq 1}$. Let $\| \ \|_p \colon \mathbb{R}^n \to \mathbb{R}$ be given by

$$\|a_1, \ldots, a_n\|_p = \big(|a_1|^p + \cdots + |a_n|^p\big)^{\frac{1}{p}}.$$

Show that $(\mathbb{R}^n, \| \ \|_p)$ is a Banach space.

5. (The space $(\mathbb{R}^n, \| \|_\infty)$) Let $\| \|_\infty \colon \mathbb{R}^n \to \mathbb{R}$ be given by
$$\|a_1, \ldots, a_n)\|_\infty = \sup(|a_1|, \ldots, |a_n|\}.$$
Show that $(\mathbb{R}^n, \| \|_\infty)$ is a Banach space.

6. (The space $(\ell^p, \| \|_p)$) Let ℓ^p be the vector space of sequences $(a_1, a_2, \ldots)$ in $\mathbb{R}$ such that $\sum_{n\in\mathbb{Z}_{>0}} |a_n|^p < \infty$. Let
$$\|(a_1, a_2, \ldots)\|_p = \left(\sum_{n\in\mathbb{Z}_{>0}} |a_n|^p\right)^{\frac{1}{p}}.$$
Show that $(\ell^p, \| \|_p)$ is a Banach space.

7. (The space $(\ell^\infty, \| \|_\infty)$) Let ℓ^∞ be the vector space of bounded sequences $(a_1, a_2, \ldots)$ in $\mathbb{R}$ with
$$\|(a_1, a_2, \ldots)\|_\infty = \sup\{|a_i| \mid i \in \mathbb{Z}_{>0}\}.$$
Show that $(\ell^\infty, \| \|_\infty)$ is a Banach space.

8. (The space $(C_b(X, \mathbb{K}), \| \|)$) Let X be a topological space. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $C_b(X, \mathbb{K})$ be the vector space of bounded continuous functions with
$$\|f\| = \sup\{|f(x)| \mid x \in X\}.$$
Show that $(C_b(X, \mathbb{K}), \| \|)$ is a Banach space.

9. (The space $(\mathbb{C}^n, \langle, \rangle)$) Let $\langle, \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by
$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$
Show that $(\mathbb{C}^n, \langle, \rangle)$ is a Hilbert space.

10. (The space $(\ell^2, \langle, \rangle)$) Let ℓ^2 be the set of sequences $(a_1, a_2, \ldots)$ in $\mathbb{C}$ such that $\sum_{i\in\mathbb{Z}_{>0}} |a_i|^2 < \infty$. Let
$$\langle (x_1, x_2, \ldots), (y_1, y_2, \ldots)\rangle = \sum_{i\in\mathbb{Z}_{>0}} x_i \overline{y_i}.$$
Show that $(\ell^2, \langle, \rangle)$ is a Hilbert space.

11. (The space $(L^2([a, b]), \langle, \rangle)$) Carefully define the space $L^2([a, b])$ and show that if
$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt$$
then $(L^2([a, b]), \langle, \rangle)$ is a Hilbert space.

12. (the spaces ℓ^p)

    (a) Carefully define the spaces ℓ^p.

    (b) Show that $\ell^1 \subseteq \ell^2 \subseteq \ell^3$.

    (c) Show that $\ell^1 \neq \ell^2 \neq \ell^3$.

13. (The dual of ℓ^p) Let $p \in \mathbb{R}_{\geq 1}$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of the Banach space ℓ^p is ℓ^q.

14. (ℓ^p is reflexive) Let $p \in \mathbb{R}_{\geq 1}$. Show that ℓ^p is a reflexive Banach space.

15. (ℓ^p and its dual) Let $p \in \mathbb{R}_{>1}$ and define $q \in \mathbb{R}_{>1}$ by $\frac{1}{p} + \frac{1}{q} = 1$.

    (a) Define the normed vector space ℓ^p.

    (b) Show that ℓ^p is a Banach space.

    (c) Prove that the dual of ℓ^p is ℓ^q.

16. (inclusions of ℓ^p spaces) Let $p \in \mathbb{R}_{\geq 1}$ and define

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_p = \left(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\right)^{1/p}$$

    for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$.

    (a) Show that if $p \leq q$ then $\ell^p \subseteq \ell^q$.

    (b) Show that if $p \neq q$ then $\ell^p \neq \ell^q$.

17. (absolute convergence condition for completeness) Let $(V, \| \ \|)$ be a normed vector space. Show that $(V, \| \ \|)$ is a Banach space if and only if every norm absolutely convergent series is convergent.

18. (if W is complete then $B(V, W)$ is complete) Let V, W be normed vector spaces. Carefully define $B(V, W)$, the vector space of bounded linear operators with the operator norm. Show that if W is a Banach space then $B(V, W)$ is a Banach space.

26.11 Bases

1. (Separable bases have orthonormal topological bases) Let H ba Hilbert space. If H has a countable dense set C then there exists an orthonormal sequence $(a_1, a_2, \ldots)$ in H with $\overline{\mathbb{K}\text{-span}\{a_1, a_2, \ldots\}} = H$.

2. (Schauder bases are topological bases) Let $(V, \| \ \|)$ be a normed vector space. Show that if $(b_1, b_2, \ldots)$ is a Schauder basis of V then $V \subseteq \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}$.

3. (Schauder bases are total sets) Let $(V, \| \ \|)$ be a normed $\mathbb{K}$-vector space. A *total set* is a set B such that $\overline{\mathbb{K}\text{-span(B)}} = V$, where $\overline{A}$ denotes the closure of A.
   Show that a Schauder basis of V is a total set.

4. (Schauder bases give separability)
   A metric space (X, d) is *separable* if X has a countable dense set.
   Let $(V, \| \ \|)$ be a normed vector space. Show that if V has a Schauder basis then V is separable.

5. (Countable dense sets give rise to total sets) Let $(V, \| \ \|)$ be a normed $\mathbb{K}$-vector space. A *total set* is a set B such that $\overline{\mathbb{K}\text{-span(B)}} = V$, where $\overline{A}$ denotes the closure of A. Show that there exists a countable dense set in V is and only if there exists a countable total set in V.

6. (A basis of ℓ^p) Let $p \in \mathbb{R}_{>1}$. Let $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis of ℓ^p.

7. (Schauder bases for ℓ^1) Let $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis of ℓ^1.

8. Let $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is not a Schauder basis of ℓ^∞.

9. Show that ℓ^∞ is not separable.

10. Show that ℓ^∞ does not have a Schauder basis.

11. (Constructing a Schauder basis of $C_b([0, 1)])$) Let $C_b([0, 1], \mathbb{R})$ be the vector space of bounded continuous functions on $[0, 1]$.
    (a) Show that the set of polynomials is dense in the space of continuous functions on $[0, 1]$ with the supremum norm.
    (b) Show that the polynomials with rational coefficients form a countable dense set in $C_b([0, 1], \mathbb{R})$.
    (c) Show that $C_b([0, 1], \mathbb{R})$ is separable.

12. (Separable Hilbert spaces have Schauder bases) Let $(V, \langle, \rangle)$ be a separable Hilbert space. Show that V has a Schauder basis.

13. (Every separable Hilbert space is ℓ^2) Let $(H, \langle, \rangle)$ be an infinite dimensional separable Hilbert space.
    (a) Show that $H \cong \ell^2$. More precisely, show that there is an invertible linear transformation $\Phi \colon H \to \ell^2$ such that

$$\text{if } x, y \in H \quad \text{then} \quad \langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle.$$

(b) Show that ℓ^∞ is a metric space and ℓ^∞ is not separable.

14. (separable Hilbert spaces have Schauder bases) Let $(V, \langle,\rangle)$ be a separable Hilbert space. Show that V has a Schauder basis.

15. (countable orthonormal total sets are Schauder bases) Let $(V, \langle,\rangle)$ be a Hilbert space. Assume that V has a countable orthonormal set $\{a_1, a_2, \ldots\}$ which is a total set. Show that $\{a_1, a_2, \ldots\}$ is a Schauder basis for V.

16. (Sums of closed subspaces) If W, V are closed subspaces of a Hilbert space H and $W \perp V$ then show that $W + V$ is closed.

17. (If $S^\perp = 0$ then S fills H) Let S be a subset of a Hilbert space H satisfying $S^\perp = \{0\}$. Show that $\overline{\mathbb{K}\text{-span}(S)} = H$.

26.12 Orthogonals and adjoints

1. ($^\perp$ is a Galois correspondence) Let S be a subset of V.
   (a) Show that the diagonal condition shows that $S \cap S^\perp = \emptyset$ or $S \cap S^\perp = \{0\}$.
   (b) Show that $S^\perp$ is a closed subspace of V.
   (c) Show that $(S^\perp)^\perp \supseteq S$.
   (d) Show that if $S \subseteq T$ then $T^\perp \subseteq S^\perp$.
   (e) Show that $((S^\perp)^\perp)^\perp = S^\perp$ $(((S^\perp)^\perp)^\perp \supseteq S^\perp$ and since $S \subseteq (S^\perp)^\perp$ then $((S^\perp)^\perp)^\perp \subseteq S^\perp)$.

2. (Orthogonals) Let A, B be subsets of a Hilbert space H. Show that
   (a) $A^\perp$ is a closed subspace of H
   (b) $A \cap A^\perp \subseteq \{0\}$
   (c) $A \subseteq B \Rightarrow A^\perp \supseteq B^\perp$
   (d) $A \subseteq A^{\perp\perp}$.
   (e) If W is a subspace of H then W is closed if and only if $W = W^{\perp\perp}$.

3. (Gram-Schmidt works) Let $(V, \langle,\rangle)$ be an inner product space. Show that the Gram-Schmidt process produces an orthonormal basis of V.

4. (If $S^\perp = 0$ then S fills H) Let S be a subset of a Hilbert space H satisfying $S^\perp = \{0\}$. Show that $\overline{\mathbb{K}\text{-span}(S)} = H$.

5. (Expansions in orthonormal sequences) Let H be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H.

(a) Let $x \in H$. Show that $\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

(b) Show that $P(x) = \sum_{n=1}^{\infty} \langle x, a_n\rangle a_n$, exists in H.

(c) Let $W = \mathrm{span}\{a_1, a_2, \ldots\}$. With $P(x)$ as in (b), show that $P(x) \in \overline{W}$.

(d) With W as in (c) and $P(x)$ as in (b), show that $x - P(x) \in \overline{W}^{\perp}$.

6. (Fourier's orthonormal sequences)

(a) Let $L^2(\mathbb{T})$ be the set of (Lebesgue measurable) functions $f\colon [-\pi, \pi] \to \mathbb{C}$ such that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \ < \infty.$$

Prove that $L^2(\mathbb{T})$ with $\langle, \rangle\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx,$$

is a Hilbert space.

(b) Prove that setting $a_n = e^{inx}$ defines an orthonormal sequence $(a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots)$ in $L^2(\mathbb{T})$.

(c) Expand the function $f(x) = x^2$ in terms of the orthonormal sequence of (b).

(d) Evaluate the expansion in (c) at $x = \pi$ to prove that

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

7. (Fourier decomposition) Show that the functions

$$e_m(t) = \frac{1}{\sqrt{2\pi}} e^{imt}, \qquad \text{for } m \in \mathbb{Z},$$

form an orthonormal basis of $L^2([0, 2\pi])$.

8. (Legendre polynomials) Let H be the Hilbert space $L^2[-1, 1]$. Show that Gram Schmidt applied to the total set $\{1, t, t^2, t^3, \ldots\}$ yields an orthonormal basis which is the sequence of Legendre polynomials given by

$$L_k(t) = c_k \frac{d^k}{dt^k}(t^2 - 1)^k, k = 1, 2, 3, \ldots$$

where the c_k are determined by requiring the polynomials to have unit length. In particular, show that the polynomials are orthogonal for any choice of c_k. (You don't need to compute the c_k).

9. (Parseval's identities) Let $S = \{e_1, e_2, \dots\}$ be a countable orthonormal basis for a separable Hilbert space H. Prove that

   (a) $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ (Fourier series),

   (b) $\|x\|^2 = \sum_{n=1}^{\infty} (\langle x, e_n \rangle)^2$ (Parseval's identity for norms),

   (c) $\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$ (Parseval's identity for inner products).

10. (the order of the sum doesn't matter) Let $(V, \langle , \rangle)$ be a Hilbert space. Let $(a_1, a_2, \dots)$ be an orthonormal sequence in V. Let $x \in V$. Show that

$$P(x) = \sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n \qquad \text{is independent of the order of the terms in the sum.}$$

11. (Bessel's inequality) Let $(V, \langle , \rangle)$ be a Hilbert space. Let $(a_1, a_2, \dots)$ be an orthonormal sequence in V. Let $x \in V$. Show that

$$\sum_{n \in \mathbb{Z}_{>0}} |\langle x, a_n \rangle|^2 \le \|x\|^2.$$

12. (The Hilbert space $\ell^2(S)$) Let S be an arbitrary set. Let $\ell^2(S)$ be the set of functions $f \colon S \to \mathbb{C}$ such that $f(s) \neq 0$ for countably many $s \in S$ and such that the series $\Sigma_{\{s \in S\}} |f(s)|^2$ converges. Define $\langle , \rangle \colon \ell^2(S) \times \ell^2(S) \to \mathbb{C}$ by

$$\langle f, g \rangle = \sum_{s \in S} \langle f(s), g(s) \rangle.$$

   Prove that

   (a) $\ell^2(S)$ is a Hilbert space.

   (b) $\ell^2 = \ell^2(\mathbb{Z}_{>0})$

   (c) Every Hilbert space with an orthonormal basis S is isometric to $l^2(S)$.

   (Hint: define functions

$$f_e : S \to \mathbb{C} \qquad \text{by} \qquad f_e(e') = \begin{cases} 1, & \text{if } e' = e, \\ 0, & \text{if } e' \neq e. \end{cases}$$

   Show that $B = \{f_e \mid e \in S\}$ is an orthonormal basis for $l^2(S)$ and the bijection $e \to f_e$ extends to an isometry $H \to l^2(S)$.)

13. (Constructing the orthogonal projection) Let W be a subspace of a Hilbert space H which admits an orthogonal projection P. Show;

(a) $P^2 = P$
(b) dist $(x, W) = \|x - Px\|$, ie Px is the closest point to x in W.

14. (projections as idempotents) Show that $H \cong W \oplus W^\perp$ if and only if there is a a linear transformation $P\colon H \to W$ such that $P^2 = P$ and $\mathrm{Res}^H_W(P) = \mathrm{id}_W$. Then

$$H = \mathrm{id}_H(H) = P(H) + (\mathrm{id}_H - P)(H) = W \oplus W^\perp.$$

15. (closed subspaces or Hilbert spaces have complements) Let $(V, \langle,\rangle)$ be a Hilbert space and let W be a closed subspace of V. Show that $V = W \oplus W^\perp$.

16. (Orthogonal projections are unique) Let $(V, \langle,\rangle)$ be an inner product space. Let W be a vector subspace of V. Show that if W admits an orthogonal projection P then P is unique.

17. (If an orthogonal projection onto W exists then W is closed) Let $(V, \langle,\rangle)$ be a Hilbert space. Let W be a vector subspace of V. Show that if there is an orthogonal projection P onto W then W is closed.

18. (Orthogonal projection in terms of an orthonormal sequence) Let $(V, \langle,\rangle)$ be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in V. Let $W = \mathrm{span}\{a_1, a_2, \ldots\}$ and let $M = \overline{W}$ be the closure of W. Show that $P\colon V \to V$ given by

$$P(x) = \sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n \qquad \text{is an orthonormal projection onto } W.$$

19. Let V be a Banach space and let V'' be the dual of the dual of V. Define $\varphi\colon V \to V''$ by

$$(\varphi(x))(f) = f(x), \qquad \text{for } f \in V'.$$

Show that φ is injective.

20. (surjectivity in the Riesz representation theorem) Let H be a Hilbert space and let $f\colon H \to \mathbb{C}$ be a bounded linear functional. Show that there exists a unique $a \in H$ such that if $x \in H$ then $f(x) = \langle x, a \rangle$.

21. (dual of a Hilbert space) Let H be a Hilbert space and let $a \in H$. Prove that the $f\colon H \to \mathbb{K}$ given by

$$f(x) = \langle x, a \rangle \qquad \text{is a bounded linear functional} \quad \text{and} \quad \|f\| = \|a\|.$$

22. (The Riesz representation theorem)

(a) State the Riesz representation theorem for bounded linear functionals on a Hilbert space H.

(b) Let ℓ^2 be the Hilbert space of real sequences $\{(a_1, a_2, a_3, \dots) \mid \sum_{i=1}^{\infty} a_i^2 < \infty\}$ with inner product
$$\langle (a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots) \rangle = \sum i = 1^{\infty} a_i b_i.$$
Define $\psi\colon \ell^2 \to \mathbb{R}$ by
$$\psi(a_1, a_2, a_3, \dots) = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$
Find a vector $v \in \ell^2$ such that
$$\text{if } a = (a_1, a_2, a_3, \dots) \in \ell^2 \quad \text{then} \quad \psi(a) = \langle a, v \rangle.$$
Use this to compute $\|\psi\|$.

(c) Let $X = C[0, 1]$ be the Banach space of continuous functions $f\colon [0, 1] \to \mathbb{R}$ with the supremum norm.
$$\text{Define } \phi\colon X \to \mathbb{R} \quad \text{by} \quad \phi(x) = x(0).$$
Prove that ϕ is a bounded linear functional.

23. (The Riesz representation theorem) Suppose that $(H, < \cdot >)$ is a real Hilbert space.

(a) Prove that the functional $f : H \to \mathbb{R}$ given by $f(x) =< x, v >$ is a bounded linear operator, where v is a fixed element of H. Compute $\|f\|$ for this functional.

(b) State the Riesz representation theorem.

(c) Suppose that $T : V \to W$ is a bounded linear operator between Banach spaces V, W. Use the Riesz representation theorem to give the construction of an adjoint operator to T. Prove that the adjoint operator is uniquely defined by your construction and is a linear operator. (You don't have to prove that the adjoint operator is bounded).

24. (The Riesz representation theorem) Suppose that $(H, < \cdot >)$ is a real Hilbert space.

(a) Prove that the functional $f : H \to \mathbb{R}$ given by $f(x) =< x, v >$ is a bounded linear operator, where v is a fixed element of H. Compute $\|f\|$ for this functional.

(b) State the Riesz representation theorem.

(c) Explain why the Riesz theorem gives an isometry between the space H^* of bounded linear functionals on a Hilbert space H and H itself, where the norm on an element $f \in H^*$ is the operator norm $\|f\|$.

25. (Inner product characterization of the adjoint operator) Let H_1 and H_2 be Hilbert spaces. Let $T\colon H_1 \to H_2$ be a bounded linear operator. Then $T^*\colon H_2 \to H_1$ is given by
$$\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1, \quad \text{for } x \in H_1 \text{ and } y \in H_2.$$

26. (adjoints in Hilbert spaces) Let H_1 and H_2 be Hilbert spaces and let $T\colon H_1 \to H_2$ be a bounded linear transformation.

(a) Show that there exists a unique function $T^*: H2 \to H1$ such that if $x \in H_1$ and $y \in H_2$ then $\langle Tx, y\rangle_2 = \langle xT^*y\rangle_1$.

(b) Show that T^* is a linear transformation.

(c) Show that T^* is bounded.

(d) Show that $\|T^*\| = \|T\|$.

27. (The Hilbert space ℓ^2) Let $(\ell^2, \langle,\rangle)$ denote the Hilbert space of sequences $(a_1, a_2, \dots)$, satisfying $\sum_{n=1}^{\infty} |a_n|^2$ is convergent. The inner product is defined by

$$\langle (a_1, a_2, \dots), (b_1, b_2, \dots)\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$$

Let $T: \ell^2 \to \ell^2$ be a linear transformation.

(a) Define what it means for a set to be a Schauder basis for a separable Banach or Hilbert space.

You may assume that l^2 has a Schauder basis $\mathcal{S} = \{e_1, e_2 \dots\}$ where $e_1 = (1, 0, 0 \dots), e_2 = (0, 1, 0, \dots), \dots$.

(b) Show that T is a bounded linear operator if and only if the sequence $\|T(e_1)\|, \|T(e_2)\|, \dots$ is bounded.

(c) If $Te_j = \sum_{n=1}^{\infty} c_{jn} e_n$, give a condition on the coefficients c_{jn} which is necessary and sufficient for T to be self adjoint. Give reasons for your answer.

28. (adjoint of the adjoint for Hilbert spaces) Let H_1 and H_2 be Hilbert spaces and let $T: H_1 \to H_2$ be a bounded linear transformation. Show that $(T^*)^* = T$.

29. Let $T: \mathbb{C}^m \to \mathbb{C}^n$ be a linear transformation. Let A be the matrix of T and let $A^* = \overline{A}^t$.

(a) Show that the matrix of T^* is A^*.

(b) Show that $\|T\| = \sqrt{\gamma}$, where γ is the largest eigenvalue of A^*A.

30. Let V and W be normed vector spaces and let $T: V \to W$ be a bounded linear operator. Show that T^*T is self adjoint and positive.

31. Let V and W be Banach spaces. Let V' be the dual of V and let W' be the dual of W. Let $T: V \to W$ be a bounded linear operator. Define $T^*: W' \to V'$ by

$$T^*f = f \circ T.$$

Show that T^* is a well defined bounded linear operator.

32. Let V and W be reflexive Banach spaces and let $T: V \to W$ be a bounded linear operator.

(a) Show that T is transformed to T^{**} by the isomorphisms $V \cong V''$ and $W \cong W''$.

(b) Show that $\|T^*\| = \|T\|$.

(c) Show that if V and W are Hilbert spaces then T is transformed to T^* by the natural isomorphisms $V \cong V'$ and $W \cong W'$.

33. Let R and L be the left and right shift operators in the normed space ℓ^p. So

$$R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, \dots) \qquad \text{and} \qquad L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots).$$

(a) Show that R, L are bounded linear operators.

(b) Find the norms $\|L\|, \|R\|$.

(c) For the case $p = 2$ find the adjoints of the shift operators $R\colon \ell^2 \to \ell^2$ and $L\colon \ell^2 \to \ell^2$.

34. Prove the following facts about adjoints of bounded linear operators on Hilbert spaces.

(a) $(T + S)^* = T^* + S^*$

(b) $(TS)^* = S^*T^*$

(c) $(\lambda T)^* = \overline{\lambda} T^*$

(d) $\|T^*T\| = \|T\|^2$

35. Let S_n be a sequence of self adjoint operators on a Hilbert space H which converge pointwise to a bounded linear operator S. Show that S is self adjoint.

36. Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero compact self adjoint operator. Show that there exists an orthonormal basis of eigenvectors for H.

37. Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero compact self adjoint operator. Let Λ be the set of eigenvalues of T. If $\mu \in \Lambda$ let $P(\mu)$ be the orthogonal projection onto the subspace X_μ of eigenvectors with eigenvalue μ. Show that if $x \in H$ then $Tx = \sum_{\mu \in \Lambda} \mu P(\mu) x$.

38. If T is a positive operator on a Hilbert space H prove that T^n is positive for all $n \geq 1$.

39. Prove that if T is a positive operator then every eigenvalue of T is non-negative.

40. Let P be an orthogonal projection on a Hilbert space H. Prove that P is self adjoint, positive and $I - P$ is positive.

26.13 Norms and bounded linear operators

1. (linear operators on finite dimensional spaces are bounded) Let V and W be normed vector spaces and let $T\colon V \to W$ be a linear transformation. Show that if V is finite dimensional then T is bounded.

2. (linear operators on infinite dimensional spaces are not necessarily bounded) Let $V = C([0,1])$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$ with norm given by
$$\|f\| = \int_0^1 |f(t)|\,dt.$$
Let $T\colon V \to \mathbb{R}$ be given by $T(f) = f(0)$.
   (a) Show that V is infinite dimensional.
   (b) Show that T is not bounded.

3. (continuous linear operators are bounded) Let V, W be normed vector spaces and let $T\colon V \to W$ be a linear transformation. Show that if T is continuous then T is bounded.

4. (bounded linear operators are uniformly continuous) Let V, W be normed vector spaces and let $T\colon V \to W$ be a linear transformation. Show that if T is bounded then T is uniformly continuous.

5. (The 0 operator and the identity operator) Let V, W be normed vector spaces. Show that the identity operator $\mathrm{id}_V\colon T \to V$ has operator norm 1 and the zero operator $0\colon V \to W$ has operator norm 0.

6. (matrix entries and the ℓ^∞ norm) Let $T : \mathbb{C}^n \to \mathbb{C}^m$ be a linear transformation with $m \times n$ matrix $A = (a_{jk})$ with respect to the standard basis. Suppose the norms on $\mathbb{C}^n$ and $\mathbb{C}^m$ are both the supremum norm. Show that
$$\|T\| = \max\left\{ \sum_{k=1}^n |a_{jk}| \mid j \in \{1,\ldots,m\} \right\}.$$

7. (matrix entries and the ℓ^1 norm) Let $T : \mathbb{C}^n \to \mathbb{C}^m$ be a linear transformation with $m \times n$ matrix $A = (a_{jk})$ with respect to the standard basis. Suppose the norms in $\mathbb{C}^n$ and $\mathbb{C}^m$ are both the ℓ^1 norms. Show that
$$\|T\| = \max\left\{ \sum_{j=1}^m |a_{jk}| \mid k \in \{1,\ldots,n\} \right\}.$$

8. (diagonal operators) Let ℓ^∞ be the vector space of bounded sequences $(a_1, a_2, \ldots)$ in $\mathbb{R}$ with norm given by
$$\|(a_1, a_2, \ldots)\| = \sup\{|a_1|, |a_2|, \ldots\}.$$
Let $(\lambda_1, \lambda_2, \ldots)$ be a bounded sequence in $\mathbb{R}$. Define $T\colon \ell^\infty \to \ell^\infty$ by
$$T(a_1, a_2, \ldots) = (\lambda_1 a_1, \lambda_2 a_2, \ldots).$$
   (a) Show that T is a well defined linear transformation.
   (b) Show that $\|T\| = \sup\{|\lambda_1|, |\lambda_2|, \ldots\}$.

9. (evaluation operator) Let $a, b \in \mathbb{R}$ with $a < b$. Let $C([a,b])$ be the Banach space of continuous functions $f\colon [a,b] \to \mathbb{R}$ with the supremum norm. Let $t_0 \in [a,b]$. Define $A\colon C([a,b]) \to \mathbb{R}$ by
$$Af = f(t_0).$$
Show that A is a bounded linear functional with $\|A\| = 1$.

10. (integral operators) Let $a, b \in \mathbb{R}$ with $a < b$. Let $k\colon [a,b] \times [a,b] \to \mathbb{C}$ be a continuous function. Let
$$C([a,b]) = \{f\colon [a,b] \to \mathbb{C} \mid f \text{ is continuous}\} \quad \text{with the supremum norm.}$$
Define $T\colon X \to X$ by
$$(Tf)(t) = \int_a^b k(t,s)x(s)ds.$$
   (a) Show that $C([a,b])$ is a Banach space.
   (b) Show that if $f \in X$ then $Tf \in X$.
   (c) Show that T is a bounded linear transformation.

11. (norms of simple integral operators) Let $a, b \in \mathbb{R}$ with $a < b$. Let $C([a,b])$ be the vector space of continuous functions $f\colon [a,b] \to \mathbb{C}$ with the supremum norm. Define $T\colon C([a,b]) \to \mathbb{C}$ by
$$Tf = \int_a^b f(t)\,dt.$$
Show that the operator norm of T is $\|T\| = b - a$.

12. (shift operators) Let $p \in \mathbb{R}_{\geq 1}$. Let R and L be the left and right shift operators on the normed vector space l^p given by
$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots), \quad \text{and} \quad L(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$$
   (a) Show that R and L are bounded linear operators.
   (b) Show that R is injective but not surjective and L is surjective but not injective.
   (c) Show that $LR = \text{id}$ but $RL \neq \text{id}$.

(d) Show that if $x \in \ell^p$ then

$$\lim_{n\to\infty} \|L^n(x)\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|L^n\| \neq 0.$$

(e) Find $\|L\|$ and $\|R\|$.

13. (a diagonal operator) Let $\{a_1, a_2, \ldots\}$ be a bounded sequence of complex numbers. Define an operator $T\colon \ell^2 \to \ell^2$ by;

$$T(b_1, b_2, \ldots) = (0, a_1b_1, a_2b_2, \ldots).$$

(a) Show that T is a bounded linear operator and find $\|T\|$.

(b) Compute the adjoint operator T^*.

(c) Show that if $T \neq 0$ then $T^*T \neq TT^*$.

(d) Find the eigenvalues of T^*.

14. (norm of the operator corresponding to an infinite matrix) Let (a_{ij}) be an infinite complex matrix, $i, j = 1, 2, \ldots$, such that if $j \in \mathbb{Z}_{>0}$ then

$$c_j = \sum_i |a_{ij}| \quad \text{converges,} \qquad \text{and} \qquad c = \sup\{c_1, c_2, \ldots\} < \infty.$$

Show that the operator $T\colon \ell^1 \to \ell^1$ defined by

$$T(b_1, b_2, \ldots) = \Big(\sum_j a_{1j}b_j,\ \sum_j a_{2j}b_j,\ \ldots \Big)$$

is a bounded linear operator and that $\|T\| = c$.

15. (shift operators) Let $(\ell^2, \langle,\rangle)$ denote the Hilbert space of sequences $(a_1, a_2, \ldots)$, with $a_i \in \mathbb{C}$, satisfying $\sum_{n=1}^\infty |a_n|^2$ is convergent. The inner product is defined by

$$\langle (a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_{n=1}^\infty a_n \overline{b_n}$$

Define operators $R\colon \ell^2 \to \ell^2$ and $L\colon \ell^2 \to \ell^2$ by

$$R((a_1, a_2, \ldots)) = (0, a_1, a_2, \ldots) \qquad \text{and} \qquad L((a_1, a_2, \ldots)) = (a_2, a_3, \ldots).$$

(a) Explain why R, L are bounded linear operators and compute their operator norms $\|R\|, \|L\|$.

(b) Define the adjoint of a linear transformation from a Hilbert space to a Hilbert space.

(c) Find the adjoints of the operators R, L. Briefly explain your answer.

16. (a compact diagonal operator on ℓ^2)

(a) Let ℓ^2 be the Hilbert space of real sequences $\{(a_1, a_2, a_3, \dots) \mid \sum_{i=1}^{\infty} a_i^2 < \infty\}$ with inner product

$$\langle (a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots)\rangle = \sum_{i=1}^{\infty} a_i b_i$$

Let $T\colon \ell^2 \to \ell^2$ be the operator

$$T(a_1, a_2, a_3, \dots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots)$$

Prove that T is a bounded linear operator.

(b) Compute the operator norm of T.

(c) Prove that T is self adjoint.

(d) Prove that T is a compact operator.

17. (The derivative operator) Let $X = C^1[0,1]$ and $Y = C[0,1]$ so that functions in X are continuously differentiable and functions in Y are continuous:

$$\begin{array}{lll} Y = C[0,1], & \text{with norm given by} & \|f\| = \sup\{|f(t)| \mid t \in [0,1]\}, \text{ and} \\ X = C^1[0,1], & \text{with norm given by} & \|f\|_0 = \|f\| + \|f'\|, \end{array}$$

where $f' = \frac{df}{dt}$. Let $D : X \to Y$ be the differentiation operator $Df = \frac{df}{dt}$.

(a) Show that $D : (X, \|\cdot\|_0) \to (Y, \|\cdot\|)$ is a bounded linear operator with $\|D\| = 1$.

(b) Show that $D : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is an unbounded linear operator. (*Hint:* Consider the sequence of elements t^n in X).

18. (Alternative formula for the norm of a bounded self adjoint operator)

(a) Carefully define the norm of a linear operator $T\colon V \to V$.

(b) Carefully define bounded linear operator and self adjoint linear operator.

(c) Let $T\colon V \to V$ be a bounded self adjont linear operator. Prove that

$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}.$$

19. (Norms of bounded self adjoint operators) Let $T\colon V \to V$ be a bounded self adjont linear operator. Then

$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}.$$

26.14 Eigenvectors of linear operators

1. (Linear operators have an eigenvector) Let V be a vector space. Let $T\colon V \to V$ be a linear operator. Show that T has a nonzero eigenvector.

2. (Not all linear operators have an eigenvector) Give an example of a vector space V and a linear transformation $T: V \to V$ which does not have a nonzero eigenvector.

3. (When do linear operators have an eigenvector?) One of the two previous questions needs to be corrected. Which one, and how should it be corrected?

4. (eigenvalues in terms of noninjectivity) Let V be a normed vector space and let $T: V \to V$ be a bounded linear operator. Show that T has an eigenvector of eigenvalue λ if and only if $\lambda - T$ is not injective.

5. (sometimes injective linear operators are also surjective, and invertible) Let T be a bounded self adjoint compact operator on a Hilbert space H. Use the spectral theorem to show that if λ is a non zero complex number so that $\lambda I - T$ is a one-to-one mapping then $\lambda I - T$ is onto and has a bounded inverse.

6. (sometimes injective linear operators are also surjective, and invertible) Let T be a bounded self adjoint compact operator on a Hilbert space H. Assume λ is a non zero complex number so that $\lambda I - T$ is an onto mapping, Use the fact that

$$\text{if } N = \ker(\lambda I - T) \text{ and } R = \overline{\mathrm{im}\,(\lambda I - T)} \text{ then } N = R^{\perp}$$

to prove that $\lambda I - T$ is one-to-one and has a bounded inverse.

7. (A converse to Fredholm's theorem?) Let H be a Hilbert space and let $T: H \to H$ be a bounded linear operator. Assume that T satisfies:

$$\text{if } \lambda \in \mathbb{K} \text{ and } \lambda \neq 0 \quad \text{then} \qquad \lambda - T \text{ is injective if and only if } \lambda - T \text{ is bijective}$$

Does it follow that T is compact?

8. (When do linear operators have eigenvectors?)
   (a) Let V be a finite dimensional normed $\mathbb{C}$-vector space and let $T: V \to V$ be a bounded linear operator. Show that T has an eigenvector.
   (b) Give an example of a $\mathbb{C}$-vector space V and a bounded linear operator $T: V \to V$ that does not have an eigenvector.
   (c) Give an example of a finite dimensional $\mathbb{R}$-vector space W and a bounded linear operator $T: W \to W$ that does not have an eigenvector.
   (d) Carefully define bounded compact self adjoint linear operator.
   (e) Sketch the proof that if V is a Hilbert space and $T: V \to V$ is a bounded compact self adjoint operator then T has an eigenvector. In your sketch, point out where the compactness of T and the fact that T is self adjoint are needed.

9. (linear operators on finite dimensional normed vector spaces) Let H be a finite dimensional Hilbert space. Let $T: H \to H$ be a linear operator. Show that

(a) T is bounded.
(b) T is compact.
(c) T has a nonzero eigenvector.

10. (bounded compact self adjoint linear operators have eigenvectors) Let H be a Hilbert space. Let $T\colon H \to H$ be a bounded compact self adjoint linear operator. Show that T has a nonzero eigenvector with eigenvalue $\|T\|$.

11. (orthonormal bases of eigenvectors) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded compact self adjoint operator. Show that there exists an orthonormal basis of eigenvectors of H.

12. (Eigenvalues of self adjoint operators are real) Let $T\colon V \to V$ be a self adjoint linear operator and let v be an eigenvector of T with eigenvalue λ. Show that $\lambda \in \mathbb{R}$.

13. (adjoints and eigenvectors) Let $V = \mathbb{C}^5$ with the standard Hermitian inner product. Let

$$T\colon V \to V \qquad \text{and} \qquad W\colon V \to V$$

be the linear transformations such that the matrices of T and W with respect to the standard basis of $V = \mathbb{C}^5$ are given by

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}$$

respectively.

(a) Compute $\|T\|$ and $\|W\|$.
(b) Let T^* be the adjoint of T and let W^* be the adjoint of W. Compute the matrices of T^* and W^* with respect to the standard basis of $\mathbb{C}^5$.
(c) Show that T and W are compact operators.
(d) Find an eigenvector of W with eigenvalue $\|W\|$.
(e) Find an orthonormal basis of V which consists of eigenvectors of W.
(f) Show that T has an eigenvector.

14. (eigenvectors)

(a) Carefully define linear operator, eigenvector and eigenvalue.
(b) Let V be a complex vector space and let $T\colon V \to V$ be a linear operator. Prove that there exists $v \in V$ with $v \neq 0$ such that v is an eigenvector of T.

(c) Use the proof of (b) to explicitly produce an eigenvector of the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 0 & 2 \\ \pi & \sqrt{7} & 0 \end{pmatrix}.$$

(d) Let V be the real vector space $\mathbb{R}^2$. Give an example of a linear transformation $T\colon V \to V$ that does not have a nonzero eigenvector.

15. (Existence of eigenvectors of bounded self adjoint linear operators) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint operator.

(a) Show that there exists $x \in H$ with $\quad \|x\| = 1 \quad$ and $\quad |\langle Tx, x\rangle| = \|T\|$.

(b) Let $x \in H$ be as in (a). Show that x is an eigenvector of T with eigenvalue $\|T\|$.

(c) Use the proof of (a) to explicitly produce an eigenvector of the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & \pi \\ -2 & \pi & 0 \end{pmatrix}.$$

16. (eigenspaces of compact linear operators) Let H be a Hilbert space.

(a) Carefully define compact linear operator.

(b) Give an example (with proof) of a bounded linear operator $T\colon \ell^2 \to \ell^2$ which is compact and a bounded linear operator $S\colon \ell^2 \to \ell^2$ which is not compact.

(c) Let $T\colon H \to H$ be a compact linear operator. Assume $\lambda \in \mathbb{C}$ and $\lambda \neq 0$ and let

$$X_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

Show that X_λ is a subspace of H and that $\dim(X_\lambda)$ is finite.

17. (eigenspaces of self adjoint operators) Let $T\colon V \to V$ be a self adjoint linear operator.

(a) Let v be an eigenvector of T with eigenvalue λ. Prove that $\lambda \in \mathbb{R}$.

(b) Let λ and γ be eigenvalues of T with $\lambda \neq \gamma$. Let

$$X_\lambda = \{v \in V \mid Tv = \lambda v\} \qquad \text{and} \qquad X_\gamma = \{v \in V \mid Tv = \gamma v\}.$$

Prove that X_λ is orthogonal to X_γ.

18. (properties of eigenspaces) Prove the following:

(a) If $T\colon H \to H$ is self adjoint and $X_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

(b) If $T\colon H \to H$ is self adjoint and $\lambda \neq \gamma$ then $X_\lambda \perp X_\gamma$.

(c) If $T\colon H \to H$ is compact operator and $\lambda \neq 0$ then X_λ is finite dimensional.

(d) If $T\colon H \to H$ is self adjoint then $W^\perp$ is a T-submodule of H.

(e) If $T\colon T \to H$ a compact operator then the restriction of T to $W^\perp$ is a compact operator.

(e) If $T\colon H \to H$ is a compact operator then $W^\perp = 0$.

19. (The sum of eigenspaces) Let
$$W = \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda.$$
Primary questions: If $W = \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda$ then
$$\text{When is } W = 0? \qquad \text{When is } W \text{ dense in } H?$$

20. (Producing an vector with $|\langle Tx, x\rangle| = \|T\|$) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint operator. Show that there exists $x \in H$ with
$$\|x\| = 1 \qquad \text{and} \qquad |\langle Tx, x\rangle| = \|T\|.$$

21. (Producing an eigenvector of a bounded self adjoint operator) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint operator. Let $x \in H$ with
$$\|x\| = 1 \qquad \text{and} \qquad |\langle Tx, x\rangle| = \|T\|.$$
Show that x is an eigenvector of T with eigenvalue $\|T\|$.

22. (The spectral theorem in an example) Let $V = \mathbb{C}^5$ with the standard Hermitian inner product. Let $T\colon V \to V$ be the linear transformation such that the matrix of T with respect to the standard basis of $V = \mathbb{C}^3$ is given by
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 3 & 7 & 10 \end{pmatrix}$$
respectively.

(a) Show that T is a compact self adjoint operator.

(b) Find an orthonormal basis of V which consists of eigenvectors of T.

(c) Give an example of Hilbert space H and a diagonal self adjoint operator $T\colon H \to H$ which is not compact.

23. (the spectral theorem)

(a) Give the definition of a compact self adjoint linear operator $T : V \to W$ where V, W are Hilbert spaces.

(b) State the spectral expansion theorem for compact self adjoint linear operators.

(c) Prove that the sum of two compact self adjoint linear operators is compact and self adjoint. (Hint: You may use the fact that if A, B are compact subsets of a normed space, then $A + B = \{a + b : a \in A, b \in B\}$ is compact.)

24. (the spectral theorem)

    (a) Give the definition of a compact self adjoint linear operator $T : V \to W$ where V, W are Hilbert spaces.

    (b) State the spectral expansion theorem for compact self adjoint linear operators.

    (c) Explain why the eigenspace of a compact self adjoint operator corresponding to a non-zero eigenvalue must be finite dimensional.

26.15 Compact operators

1. (Compact operators have finite dimensional eigenspaces) Let $T\colon H \to H$ be a compact linear operator. Assume $\lambda \in \mathbb{K}$ and $\lambda \neq 0$ and let
$$X_\lambda = \{v \in H \mid Tv = \lambda v\}. \qquad \text{Show that} \quad \dim(X_\lambda) \text{ is finite.}$$

2. (finiteness) Show that if we think topologically then we realize that compactness for bounded linear operators is a fiiteness condition. Also, if we think linear algebraically then, if H is finite dimensional then $T\colon H \to H$ is always compact.

3. (the closed unit ball is compact if and only if it is finite dimensional) Let $(V, \|\ \|)$ be a Banach space.

    (a) Show that if V is finite dimensional then the closed unit ball in V is compact.

    (b) Show that if V is infinite dimensional then the closed unit ball in V is not compact.

4. (infinite sequences of far apart unit vectors) Let $(V, \|\ \|)$ be an infinite dimensional Banach space. Construct a sequence $(e_1, e_2, \ldots)$ of unit vectors in V such that if $i, j \in \mathbb{Z}_{>0}$ and $i \neq j$ then $d(e_i, e_j) > \frac{1}{2}$.

5. (alternative definition of a compact operator) Let X be a normed vector space and let $B = \{x \in X \mid \|x\| \leq 1\}$. Let $T\colon X \to X$ be a linear operator. Show that T is compact if and only if $\overline{T(B)}$ is compact.

6. (images of compact operators) Let X be a normed vector space and let A be a bounded subset of X. Let $T\colon X \to X$ be a compact operator. Show that $\overline{T(A)}$ is compact.

7. (linear operators on finite dimensional spaces are compact) Let X be a finite dimensional normed vector space and let $T\colon X \to X$ be a linear transformation. Show that T is a compact operator.

8. (integral operators on closed intervals are compact) Let $X = C([a,b])$ be the space of continuous functions $f\colon [a,b] \to \mathbb{R}$ with the supremum norm. Let $k\colon [a,b]\times[a,b] \to \mathbb{C}$ be a continuous function and define $T\colon X \to X$ by

$$(Tf)(t) = \int_a^b k(t,s)f(s)\,ds.$$

Show that T is a compact operator.

9. (computing the norm by inner products) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint operator. Show that

$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid x \in H, \|x\| = 1\}.$$

10. (characterising eigenvectors of the largest eigenvalue) Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero compact self adjoint operator.

    (a) Show that there exists an eigenvalue λ of T such that $|\lambda| = \|T\|$.

    (b) Show that if v is an eigenvector of T with eigenvalue λ such that $|\lambda| = \|T\|$ then v is a solution of the extremal problem

$$\max\{\langle Tu, u\rangle \mid u \in H, \|u\| = 1\}.$$

11. (Limits of eigenvalues of compact operators) Let H be an infinite dimensional Hilbert space. Let $T\colon H \to H$ be a bounded self adjoint compact operator. Show that the eigenvalues of T form a sequence converging to 0.

12. (eigenvectors from differential operators) Let $a, b \in \mathbb{R}$ with $a < b$. Let $\lambda \in \mathbb{R}$ and let $p\colon [a,b] \to \mathbb{R}_{>0}$ and $q\colon [a,b] \to \mathbb{R}$ with $p \in C'([a,b])$ and $q \in C^2([a,b])$. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $(a_1, a_2) \neq (0,0)$ and $(b_1, b_2) \neq (0,0)$. Let $L\colon C^2([a,b]) \to C([a,b])$ be given by

$$Ly = (-py')' + qy.$$

Let $u, v \in C^2([a,b])$ such that

$$Lu = 0, \quad Lv = 0, \quad a_1u(a) + a_2u'(a) = 0, \quad \text{and} \quad b_1v(b) + b_2v'(b) = 0.$$

Let $G\colon [a,b]\times[a,b] \to \mathbb{R}$ be given by

$$G(s,t) = \begin{cases} v(t)u(s), & \text{if } s \le t, \\ u(t)v(s), & \text{if } t \le s. \end{cases}$$

Define $T\colon L^2([a,b]) \to L^2([a,b])$ by

$$(Tf)(t) = \int_a^b G(t,s)f(s)ds, \qquad \text{for } t \in [a,b].$$

(a) Show that the eigenvalues of T are nonzero and each eigenvector f satisfies $a_1 f(a) + a_2 f'(a) = 0$ and $b_1 f(b) + b_2 f'(b) = 0$.

(b) Show that f is an eigenvector of T with eigenvalue μ if and only if f is an eigenvector of L with eigenvalue $\frac{1}{\mu}$.

(c) Show that L has a sequence of eigenvalues $\lambda \to \infty$, each eigenspace of L is one dimensional and there is an orthonormal basis of $L^2([a,b])$ of eigenvectors of L.

13. (Fourier modes as eigenvectors) Let $G\colon [0,\pi] \times [0,\pi] \to \mathbb{R}$ be given by

$$G(t,s) = \begin{cases} t - \frac{st}{\pi}, \text{if } s \le t, \\ s - \frac{st}{\pi}, \text{if } s \ge t, \end{cases}$$

and let $T\colon L^2([0,\pi]) \to L^2([0,\pi])$ be given by

$$(Tf)(t) = \int_0^\pi G(t,s) f(s) ds, \qquad \text{for } t \in [0,\pi].$$

(a) Show that T has eigenvalues $\lambda_n = n^2$, $n \in \mathbb{Z}_{n>0}$, and corresponding eigenvectors $s_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt)$.

(b) Show that the functions $s_n(t) = \sqrt{2\pi} \sin(nt)$, $n \in \mathbb{Z}_{>0}$ form an orthonormal basis of $L^2([0,\pi])$.

(c) Show that the functions $s_n(t)$ are the solutions to the Sturm Liouville system in the last example $y'' + \lambda y = 0$ on $[0,\pi]$ with the boundary conditions $y(0) = y(\pi) = 0$.