17 Number systems

17.1 The number systems $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{R}((t))$

17.1.1 The real numbers

The real numbers $\mathbb{R}$ is the set of decimal expansions.

The *real numbers* $\mathbb{R}$ contain the *integers* $\mathbb{Z}$.

$$\begin{aligned}
\mathbb{R} &= \left\{ \pm(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + a_{-\ell+2}\left(\tfrac{1}{10}\right)^{-\ell+2} + \cdots) \mid \ell \in \mathbb{Z},\ a_j \in \frac{\mathbb{Z}}{10\mathbb{Z}} \right\} \\
&\cup\!| \\
\mathbb{Z} &= \left\{ \pm(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots + a_{-1}\left(\tfrac{1}{10}\right) + a_0) \mid \ell \in \mathbb{Z}_{\geq 0},\ a_j \in \frac{\mathbb{Z}}{10\mathbb{Z}} \right\}
\end{aligned}$$

where $\frac{\mathbb{Z}}{10\mathbb{Z}} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the addition and multiplication in $\mathbb{R}$ are compatible with the addition and multiplication in $\mathbb{Z}$.

17.1.2 The p-adic numbers

Let $p \in \mathbb{Z}_{>0}$. The *p-adic numbers* $\mathbb{Q}_p$ contain the *p-adic integers* $\mathbb{Z}_p$ and the *nonnegative integers* $\mathbb{Z}_{\geq 0}$.

$$\begin{aligned}
\mathbb{Q}_p &= \left\{ a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + a_{-\ell+2}p^{-\ell+2} + \cdots \;\middle|\; \ell \in \mathbb{Z},\ a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \right\} \\
&\cup\!| \\
\mathbb{Z}_p &= \left\{ a_0p^0 + a_1p^1 + a_2p^2 + \cdots \;\middle|\; a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \right\} \\
&\cup\!| \\
\mathbb{Z}_{\geq 0} &= \left\{ a_0p^0 + a_1p^1 + a_2p^2 + \cdots \;\middle|\; a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \text{ and all but a finite number of the } a_j \text{ are } 0 \right\},
\end{aligned}$$

where $\frac{\mathbb{Z}}{p\mathbb{Z}} = \{0, 1, 2, \ldots, p-2, p-1\}$ and the addition and multiplication in $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are compatible with the addition and multiplication in $\mathbb{Z}$.

17.1.3 Extended polynomials

Let t be a variable.

The *rational functions* $\mathbb{R}((t))$ contain the *formal power series* $\mathbb{R}[[t]]$ and the *polynomials* $\mathbb{R}[t]$.

$$\begin{aligned}
\mathbb{R}((t)) &= \left\{ a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + a_{-\ell+2}t^{-\ell+2} + \cdots \mid \ell \in \mathbb{Z},\ a_j \in \mathbb{R} \right\} \\
&\cup\!| \\
\mathbb{R}[[t]] &= \left\{ a_0t^0 + a_1t^1 + a_2t^2 + \cdots \mid a_j \in \mathbb{R} \right\} \\
&\cup\!| \\
\mathbb{R}[t] &= \left\{ a_0t^0 + a_1t^1 + a_2t^2 + \cdots \mid a_j \in \mathbb{R} \text{ and all but a finite number of the } a_j \text{ are } 0 \right\},
\end{aligned}$$

where $\mathbb{R}$ is the real numbers and the addition and multiplication in $\mathbb{R}((t))$ and $\mathbb{R}[[t]]$ are compatible with the addition and multiplication in $\mathbb{R}$.

17.1.4 Some examples to check.

In $\mathbb{R}$,

$$\begin{aligned}
\tfrac{1}{2} &= .5000000\ldots = 5\cdot 10^{-1} + 0\cdot 10^{-2} + 0\cdot 10^{-3} + \cdots\,, \\
-1 &= -(1\cdot 10^{0} + 0\cdot 10^{-1} + 0\cdot 10^{-2} + \cdots), \\
\pi &= 3.1415926\ldots = 3\cdot 10^{0} + 1\cdot 10^{-1} + 4\cdot 10^{-2} + \cdots\,, \\
1 &= 1.00000.... = 1\cdot 10^{0} + 0\cdot 10^{-1} + 0\cdot 10^{-2} + \cdots \\
&= 0.999999 = 9\cdot 10^{-1} + 9\cdot 10^{-2} + 9\cdot 10^{-3} + 9\cdot 10^{-4} + \cdots\,.
\end{aligned}$$

In $\mathbb{Q}_7$,

$$\begin{aligned}
888 &= 6 + 0\cdot 7 + 4\cdot 7^2 + 1\cdot 7^3 + 0\cdot 7^4 + 0\cdot 7^5 + 0\cdot 7^6 + \cdots\,, \\
-\tfrac{1}{6} &= \frac{1}{1-7} = 1 + 1\cdot 7 + 1\cdot 7^2 + 1\cdot 7^3 + 1\cdot 7^4 + \cdots\,, \\
-1 &= 6\cdot\left(-\tfrac{1}{6}\right) = 6 + 6\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + 6\cdot 7^4 + \cdots\,, \\
\tfrac{1}{2} &= 1 + 3\cdot\left(-\tfrac{1}{6}\right) = 4 + 3\cdot 7 + 3\cdot 7^2 + 3\cdot 7^3 + 3\cdot 7^4 + \cdots\,, \\
-6 &= 1 + 7\cdot(-1) = 1 + 6\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + 6\cdot 7^4 + \cdots\,,
\end{aligned}$$

In $\mathbb{R}((t))$,

$$\begin{aligned}
\frac{1}{1-t} &= 1 + t + t^2 + t^3 + t^4 + \cdots\,, \\
e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\,, \\
\sin t &= t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\,, \\
\frac{1}{t^3(1-t)} &= t^{-3} + t^{-2} + t^{-1} + t + t^2 + \cdots\,.
\end{aligned}$$

17.1.5 $\mathbb{R}$ and $\mathbb{Q}_p$ and $\mathbb{R}((t))$ are metric spaces

Fix a number $e \in \mathbb{R}_{>0}$.
If $x, y \in \mathbb{R}$ the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_{1/10}(y-x)}, \quad \text{where}$$

$$\mathrm{val}_{1/10}\big(\pm\,(a_\ell\left(\tfrac{1}{10}\right)^{\ell} + a_{\ell-1}\left(\tfrac{1}{10}\right)^{\ell+1} + a_{l-2}\left(\tfrac{1}{10}\right)^{\ell+2} + \cdots)\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.
If $x, y \in \mathbb{Q}_p$ then the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_p(y-x)}, \quad \text{where} \quad \mathrm{val}_p\big(a_\ell p^\ell + a_{\ell+1}p^{\ell+1} + a_{\ell+2}p^{\ell+2} + \cdots\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.
If $x, y \in \mathbb{R}((t))$ then the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_t(y-x)} \quad \text{where} \quad \mathrm{val}_t\big(a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + a_{\ell+2}t^{\ell+2} + \cdots\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.

17.2 The number systems $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$

17.2.1 The nonnegative integers $\mathbb{Z}_{\geq 0}$

The *positive integers* is the set

$$\mathbb{Z}_{>0} = \{1, 1+1, 1+1+1, 1+1+1+1, \ldots\}$$

with *addition* given by concatenation so that, for example,

$$(1+1+1) + (1+1+1+1) = 1+1+1+1+1+1+1.$$

The positive integers are often written as

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$$

is the set of *nonnegative integers* with addition determined by the addition in $\mathbb{Z}_{>0}$ and the condition

$$\text{if} \quad x \in \mathbb{Z}_{\geq 0} \quad \text{then} \quad 0 + x = x \quad \text{and} \quad x + 0 = x.$$

Define a relation on $\mathbb{Z}_{\geq 0}$ by

$$x \leq y \qquad \text{if there exists } n \in \mathbb{Z}_{\geq 0} \text{ such that } x + n = y.$$

17.2.2 The nonnegative rational numbers $\mathbb{Q}_{\geq 0}$

The *nonnegative rational numbers* is the set

$$\mathbb{Q}_{\geq 0} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}_{\geq 0},\ b \neq 0 \right\} \qquad \text{with} \qquad \frac{a}{b} = \frac{c}{d} \quad \text{if} \quad ad = bc,$$

and with addition and multiplication given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Define a relation on $\mathbb{Q}_{\geq 0}$ by

$$x \leq y \qquad \text{if there exists } a \in \mathbb{Q}_{\geq 0} \text{ such that } x + a = y.$$

If $x, y \in \mathbb{Q}_{\geq 0}$ define

$$d(x, y) = a \qquad \text{where } a \in \mathbb{Q}_{\geq 0} \text{ is such that } x + a = y \text{ or } y + a = x.$$

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, 10^{-3}, \ldots\}$ and let $\epsilon \in \mathbb{E}$. The *ϵ-diagonal* in $\mathbb{Q}_{>0}$ is

$$B_\epsilon = \{(x, y) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \mid d(x, y) < \epsilon\}.$$

Let $a \in \mathbb{Q}_{\geq 0}$ and $\epsilon \in \mathbb{E}$. The *ϵ-ball at a* is

$$B_\epsilon(a) = (a - \epsilon, a + \epsilon) = \{x \in \mathbb{Q}_{\geq 0} \mid a - \epsilon < x < a + \epsilon\}.$$

Let $\mathcal{B} = \{B_\epsilon(a) \mid \epsilon \in \mathbb{E}, a \in \mathbb{Q}_{\geq 0}\}$.

$U \subseteq \mathbb{Q}_{\geq 0}$ is an *open set in* $\mathbb{Q}_{\geq 0}$ if $\quad$ there exists $\mathcal{S} \subseteq \mathcal{B}$ such that $\quad U = \bigcup_{B \in \mathcal{S}} B$.

$N \subseteq \mathbb{Q}_{\geq 0}$ is a *neighborhood of* x if $\quad$ there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\quad N \supseteq B_\epsilon(x)$.

The *topology on* $\mathbb{Q}_{\geq 0}$ is the collection of open sets of $\mathbb{Q}_{\geq 0}$.

17.2.3 The nonnegative real numbers $\mathbb{R}_{\ge 0}$

The *nonnegative real numbers* is the set of decimal expansions

$$\mathbb{R}_{\ge 0} = \{z.d_1d_2d_3\ldots \mid z \in \mathbb{Z}_{\ge 0}, d_i \in \{0,\ldots,9\}\}$$

with a condition that $z.9999\ldots = (z+1).0000$ if $z \in \mathbb{Z}_{\ge 0}$, and

$$z.d_1\ldots d_{k+1}d_k9999\ldots = z.d_1\ldots d_{k+1}(d_k+1)000\ldots, \qquad \text{if } z \in \mathbb{Z}_{\ge 0} \text{ and } d_k \ne 9.$$

For example $0.9999\ldots = 1.0000\ldots$.

Identify a nonnegative real number $a = z.d_1d_2d_3\ldots$ with a series

$$a = z + \sum_{k\in\mathbb{Z}_{>0}} d_k\big(\tfrac{1}{10}\big)^k \qquad \text{which is really a notation for the sequence} \qquad (z, z+s_1, z+s_2, \ldots),$$

where

$$s_1 = d_1\tfrac{1}{10}, \quad s_2 = d_1\tfrac{1}{10} + d_2\big(\tfrac{1}{10}\big)^2, \quad s_3 = d_1\tfrac{1}{10} + d_2\big(\tfrac{1}{10}\big)^2 + d_3\big(\tfrac{1}{10}\big)^3, \quad \ldots.$$

Thus a decimal expansion is really a series, which is really a sequence of elements of $\mathbb{Q}_{\ge 0}$. The sequence

$$a = (a_1, a_2, \ldots) = (z+s_1, z+s_2, \ldots) \qquad \text{satisfies} \qquad \begin{matrix}\text{if } k \in \mathbb{Z}_{>0} \text{ and } n, m \in \mathbb{Z}_{\ge(k+1)} \\ \text{then } d(a_m, a_n) < 10^{-k}.\end{matrix}$$

In order to describe the addition and multiplication on $\mathbb{R}_{\ge 0}$, consider

$$\widehat{\mathbb{Q}}_{\ge 0} = \{a = (a_1, a_2, \ldots) \mid a_i \in \mathbb{Q}_{\ge 0} \text{ and } (a_1, a_2, \ldots) \text{ is Cauchy}\}$$

where a sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\ge 0}$ is *Cauchy* if it satisfies

$$\text{if } k \in \mathbb{Z}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \qquad \text{if } m, n \in \mathbb{Z}_{>N} \text{ then } d(a_m, a_n) < 10^{-k}.$$

Define $(a_1, a_2, \ldots) = (b_1, b_2, \ldots)$ if the sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ satisfy

$$\text{if } k \in \mathbb{Z}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\ge N} \text{ then } d(a_n, b_n) < 10^{-k}.$$

An equivalent way to say this is that $(a_1, a_2, \ldots) = (b_1, b_2, \ldots)$ if $\lim_{n\to\infty} d(a_n, b_n) = 0$.

With these definitions, then define addition and multiplication on $\widehat{\mathbb{Q}}_{\ge 0}$ by

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1+b_1, a_2+b_2, \ldots) \qquad \text{and} \qquad \text{(Rge0plusdefn)}$$

$$(a_1, a_2, \ldots) \cdot (b_1, b_2, \ldots) = (a_1b_1, a_2b_2, \ldots), \qquad \text{(Rge0multdefn)}$$

and define $(a_1, a_2, \ldots) < (b_1, b_2, \ldots)$

$$\text{if there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\ge N} \text{ then } a_n < b_n. \qquad \text{(Rge0orderdefn)}$$

The point is that $\mathbb{R}_{\ge 0}$ is *the same* as $\widehat{\mathbb{Q}}_{\ge 0}$: If

$$a = (a_1, a_2, \ldots) \in \widehat{\mathbb{Q}}_{\ge 0} \qquad \text{then let } z = \lfloor a_1 \rfloor \text{ and } d_k = \lfloor 10^k a_{N_k} \rfloor \bmod 10,$$

where, if $k \in \mathbb{Z}_{>0}$ then $N_k \in \mathbb{Z}_{>0}$ is such that if $m, n \in \mathbb{Z}_{\ge N_k}$ then $d(a_m, a_n) < 10^{-k}$. This produces a decimal expansion $z.d_1d_2d_3\ldots$ such that the corresponding sequence is equal to a. So, a decimal expansion is a Cauchy sequence and a Cauchy sequence is a decimal expansion.

Regarding $\mathbb{R}_{\ge 0}$ as $\widehat{\mathbb{Q}}_{\ge 0}$, define a function $\iota\colon \mathbb{Q}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$\iota(x) = (x, x, x, \ldots), \qquad \text{which is a Cauchy sequence in } \mathbb{Q}_{\ge 0}.$$

Define $d\colon \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$d(x, y) = z, \quad \text{where } z \in \mathbb{R}_{\ge 0} \text{ is such that } x + z = y \text{ or } y + z = x.$$

In terms of decimal expansions, the order relation on $\mathbb{R}_{\ge 0}$ is given by

$$x \le y \qquad \text{if } x \text{ is less than or equal to } y \text{ in lexicographic order.}$$

Propositions 17.1, 17.2, 17.3, 17.4, and 17.5 are all consequences of the analogous statements for $\mathbb{Q}_{\ge 0}$, and the definitions of addition, multiplication and the order in $\mathbb{R}_{\ge 0}$ as given in (Rge0plusdefn), (Rge0multdefn), and (Rge0orderdefn).

Proposition 17.1. ($\mathbb{R}_{\ge 0}$ is a field without subtraction)
Let $0 = 0.0000\ldots$ *and* $1 = 1.0000\ldots$.

(a) *If* $x, y, z \in \mathbb{R}_{\ge 0}$ *then* $(x + y) + z = x + (y + z)$.

(b) *If* $x \in \mathbb{R}_{\ge 0}$ *then* $0 + x = x$ *and* $x + 0 = x$.

(c) *If* $x, y \in \mathbb{R}_{\ge 0}$ *then* $x + y = y + x$.

(d) *If* $x, y, z \in \mathbb{R}_{\ge 0}$ *then* $(xy)z = x(yz)$.

(e) *If* $x \in \mathbb{R}_{\ge 0}$ *then* $1 \cdot x = x$ *and* $x \cdot 1 = x$.

(f) *If* $x \in \mathbb{R}_{\ge 0}$ *and* $x \ne 0$ *then there exists* $x^{-1} \in \mathbb{R}_{\ge 0}$ *such that* $x \cdot x^{-1} = 1$ *and* $x^{-1} \cdot x = 1$.

(g) *If* $x, y \in \mathbb{R}_{\ge 0}$ *then* $xy = yx$.

(h) *If* $x, y, z \in \mathbb{R}_{\ge 0}$ *then* $x(y + z) = xy + xz$.

Proposition 17.2. ($\mathbb{Q}_{\ge 0}$ inside $\mathbb{R}_{\ge 0}$)

(a) *if* $x, y \in \mathbb{Q}_{\ge 0}$ *then* $\iota(x + y) = \iota(x) + \iota(y)$.

(b) *If* $x, y \in \mathbb{Q}_{\ge 0}$ *then* $\iota(xy) = \iota(x)\iota(y)$.

(c) *If* $x, y \in \mathbb{Q}_{\ge 0}$ *then* $\iota(x) < \iota(y)$ *if and only if* $x < y$.

(d) ι *is injective.*

Proposition 17.3. *($\le$ is a total order on $\mathbb{R}_{\ge 0}$)*

(a) *If* $x \in \mathbb{R}_{\ge 0}$ *then* $x \le x$.

(b) *If* $x, y \in \mathbb{R}_{\ge 0}$ *then* $x \le y$ *or* $y \le x$.

(c) *If* $x, y \in \mathbb{R}_{\ge 0}$ *and* $x \le y$ *and* $y \le x$ *then* $x = y$.

(d) *If* $x, y, z \in \mathbb{R}_{\ge 0}$ *and* $x \le y$ *and* $y \le z$ *then* $x \le z$.

Proposition 17.4. *(addition multiplication and the order)*

(a) *If* $x, y, z \in \mathbb{R}_{\ge 0}$ *and* $x \le y$ *then* $x + z \le y + z$.

(b) *If* $x, y \in \mathbb{R}_{\ge 0}$ *then* $xy \in \mathbb{R}_{\ge 0}$.

Proposition 17.5. *(d is a metric on $\mathbb{R}_{\ge 0}$)*

(a) *Let* $x, y \in \mathbb{R}_{>0}$. *Then there exists a unique* $z \in \mathbb{R}_{\ge 0}$ *such that* $x + z = y$ *or* $y + z = x$.

(b) *If* $x \in \mathbb{R}_{\geq 0}$ *then* $d(x,x) = 0$.

(c) *If* $x, y \in \mathbb{R}_{\geq 0}$ *and* $d(x,y) = 0$ *then* $x = y$.

(d) *If* $x, y \in \mathbb{R}_{\geq 0}$ *then* $d(x,y) = d(y,x)$.

(e) *If* $x, y, z \in \mathbb{R}_{\geq 0}$ *then* $d(x,y) \leq d(x,z) + d(z,y)$.

Next are important properties of $\mathbb{R}_{\geq 0}$ which do not come so directly from analogous properties of $\mathbb{Q}_{\geq 0}$.

Proposition 17.6. *(The function* $\iota\colon \mathbb{Q}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is not surjective)*

(a) *There exists* $z \in \mathbb{R}_{\geq 0}$ *such that* $z^2 = 2$.

(b) *If* $z \in \mathbb{R}_{\geq 0}$ *and* $z^2 = 2$ *then* $z \notin \mathbb{Q}_{\geq 0}$.

Proposition 17.7. *(*$\mathbb{Q}_{\geq 0}$ *and the order on* $\mathbb{R}_{\geq 0}$*)*

(a) *If* $a, b \in \mathbb{R}_{\geq 0}$ *and* $a < b$ *then there exists* $c \in \mathbb{Q}_{\geq 0}$ *such that* $a < c < b$.

(b) *If* $a, b \in \mathbb{R}_{\geq 0}$ *and* $a < b$ *then there exists* $c \in (\mathbb{R}_{\geq 0} - \mathbb{Q}_{\geq 0})$ *such that* $a < c < b$.

Theorem 17.8. *(Archimedes' property)*
If $x, y \in \mathbb{R}_{\geq 0}$ *then there exists* $n \in \mathbb{Z}_{\geq 0}$ *such that* $y < nx$.

Theorem 17.9. *(The least upper bound property)*

(a) *If* $A \subseteq \mathbb{R}_{\geq 0}$ *and* $A \neq \emptyset$ *and* A *is bounded then* $\sup(A)$ *exists in* $\mathbb{R}_{\geq 0}$.

Proposition 17.10.

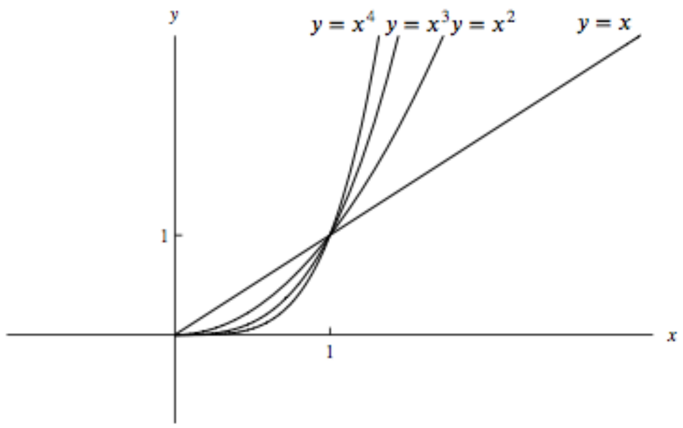
(a) *If* $(a_1, a_2, \ldots)$ *is an increasing and bounded sequence in* $\mathbb{R}_{\geq 0}$ *then* $(a_1, a_2, \ldots)$ *converges to* $\sup\{a_1, a_2, \ldots\}$.

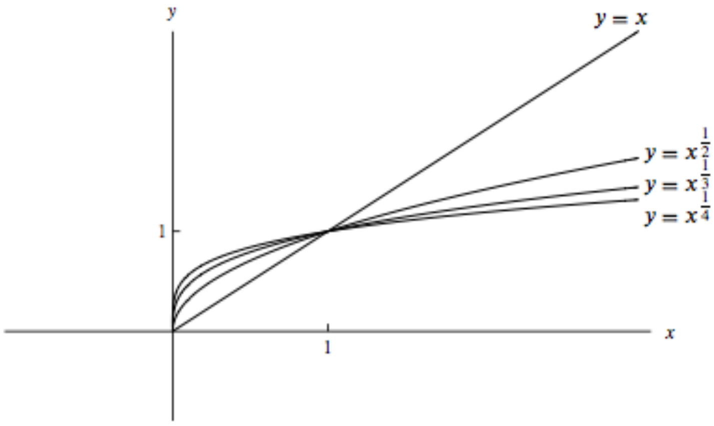
(b) $\overline{\mathbb{Q}_{\geq 0}} = \mathbb{R}_{\geq 0}$.

Theorem 17.11. *Let* $n \in \mathbb{Z}_{>0}$. *The function* $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is continuous, bijective, and satisfies*

$$\text{if } x, y \in \mathbb{R}_{\geq 0} \text{ and } x < y \quad \text{then} \quad x^n < y^n.$$

Furthermore, the inverse function $x^{\frac{1}{n}}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is continuous.*





The *standard uniformity* on $\mathbb{R}_{\geq 0}$ is

$$\mathcal{X} = \{\text{subsets of } \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \text{ which contain a set } B_\epsilon\}, \qquad \text{where}$$

$$B_\epsilon = \{(x,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mid d(x,y) < \epsilon\} \quad \text{for } \epsilon \in \{10^{-1}, 10^{-2}, \ldots\}.$$

The *standard topology on* $\mathbb{R}_{\geq 0}$ is

$$\mathcal{T} = \{\text{unions of open balls}\},$$

where the *set of open balls* is $\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \{10^{-1}, 10^{-2}, \ldots\}, x \in \mathbb{R}_{\geq 0}\}$ and

$$B_\epsilon(x) = \{y \in \mathbb{R}_{\geq 0} \mid d(x,y) < \epsilon\} \qquad \text{is the } \epsilon\text{-ball at } x.$$

Proposition 17.12. *(Topological properties of $\mathbb{R}_{\geq 0}$)*

(a) $\mathbb{R}_{\geq 0}$ is a Hausdorff topological space.

(b) $\mathbb{R}_{\geq 0}$ is a complete metric space.

(c) $\mathbb{R}_{\geq 0}$ is locally compact.

(d) $\mathbb{R}_{\geq 0}$ is not compact.

An *interval* in $\mathbb{R}_{\geq 0}$ is a set $A \subseteq \mathbb{R}_{\geq 0}$ such that

$$\text{if } x, y \in A \text{ and } z \in \mathbb{R}_{\geq 0} \text{ and } x < z < y \text{ then } z \in A.$$

Theorem 17.13. *Let $A \subseteq \mathbb{R}_{\geq 0}$.*

(a) A is connected if and only if A is an interval.

(b) A is compact if and only if A is closed and bounded.

17.3 Some proofs

17.3.1 Relations between $\mathbb{Q}_{\ge 0}$ and $\mathbb{R}_{\ge 0}$

Proposition 17.14. *(The function $\iota\colon \mathbb{Q}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is not surjective)*

(a) There exists $z \in \mathbb{R}_{\ge 0}$ such that $z^2 = 2$.

(b) If $z \in \mathbb{R}_{\ge 0}$ and $z^2 = 2$ then $z \notin \mathbb{Q}_{\ge 0}$.

Proof. (Sketch)

(a) Noting that $1^2 = 1 < 2$ and $2^2 = 4 > 2$, let $z_1 = 1$.
Noting that $14^2 = 196 < 200$ and $15^2 = 225 > 200$, let $z_2 = 1.4$.
Noting that $141^2 = 19881 < 20000$ and $142^2 = 20164 > 20000$, let $z_3 = 1.41$.
In general, for $k \in \mathbb{Z}_{\ge 0}$ let $a_k \in \mathbb{Z}_{>0}$ be maximal such that $a_k^2 < 2 \cdot 10^{2k}$ and let $z_{k+1} = 10^{-k} a_k$.
Then $z = (z_1, z_2, \ldots) \in \mathbb{R}_{\ge 0}$ and $z^2 = 2$.

(b) If $z = p/q \in \mathbb{Q}_{\ge 0}$ with p/q in reduced form then $2q^2 = p^2$ which implies 2 divides p which implies 2 divides q, which is a contradiction to p/q being reduced. THIS HEAVILY USES THE FACT THAT $\mathbb{Z}$ IS A UNIQUE FACTORIZATION DOMAIN. DO YOU KNOW HOW TO PROVE THAT $\mathbb{Z}$ IS A UNIQUE FACTORIZATION DOMAIN?? □

Proposition 17.15. *($\mathbb{Q}_{\ge 0}$ and the order on $\mathbb{R}_{\ge 0}$)*

(a) If $a, b \in \mathbb{R}_{\ge 0}$ and $a < b$ then there exists $c \in \mathbb{Q}_{\ge 0}$ such that $a < c < b$.

(b) If $a, b \in \mathbb{R}_{\ge 0}$ and $a < b$ then there exists $c \in (\mathbb{R}_{\ge 0} - \mathbb{Q}_{\ge 0})$ such that $a < c < b$.

Proof.

(a) If $a, b \in \mathbb{R}_{\ge 0}$ and $a < b$.
To show: There exists $c \in \mathbb{Q}_{\ge 0}$ such that $a < c < b$.
Let $x \in \mathbb{R}_{\ge 0}$ such that $b = a + x$.
Let $k \in \mathbb{Z}_{>0}$ such that $10^{-k} < x$
(i.e. if $x = z.d_1 d_2 d_3 \ldots$ then let $n \in \mathbb{Z}_{>0}$ such that $d_n \ne 0$ and let $k = n + 1$).
Let $c = a + 10^{-k}$.
Since $a < a + 10^{-k} < a + x = b$ then $a < c < b$.

(b) Since $\sqrt{2} \in \mathbb{R}_{\ge 0} - \mathbb{Q}_{\ge 0}$ then $c \in \mathbb{R}_{\ge 0} - \mathbb{Q}_{\ge 0}$.
Let $x \in \mathbb{R}_{\ge 0}$ such that $b = a + x$.
Let $k \in \mathbb{Z}_{>0}$ such that $10^{-k} < x$
Let $c = a + 10^{-k}\frac{\sqrt{2}}{2}$.
Since $a < a + 10^{-k}\frac{\sqrt{2}}{2} < a + 10^{-k} < a + x = b$ then $a < c < b$.

□

17.3.2 Archimedes' property and the least upper bound property

Theorem 17.16. *(Archimedes' property)*
If $x, y \in \mathbb{R}_{\ge 0}$ then there exists $n \in \mathbb{Z}_{\ge 0}$ such that $y < nx$.

Proof. Assume $x, y \in \mathbb{R}_{>0}$.
To show: There exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.
Using Proposition 17.7(a), there exist $\frac{p}{q} \in \mathbb{Q}_{>0}$ and $\frac{r}{s} \in \mathbb{Q}_{>0}$ such that

$$0 < \frac{p}{q} < x \qquad \text{and} \qquad y < \frac{r}{s}.$$

Let $n \in \mathbb{Z}_{>0}$ be such that $nps > qr$.
Then

$$y < \frac{rq}{sq} < \frac{nsp}{sq} = nx.$$

□

Theorem 17.17. *(The least upper bound property)*

(a) If $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists in $\mathbb{R}_{\geq 0}$.

Proof. (Sketch) If $a = zd_1d_2d_3\ldots$ is the decimal expansion of a and $k \in \mathbb{Z}_{>0}$ then let

$$a_k = z.d_1d_2\cdots d_k \in \mathbb{Q}_{\geq 0} \qquad \text{(this is the kth element of the sequence corresponding to a).}$$

For $k \in \mathbb{Z}_{>0}$, define

$$A_k = \{a_k \mid a \in A\} \qquad \text{so that} \qquad A_k \subseteq \mathbb{Q}_{\geq 0} \quad \text{and} \qquad \mathrm{Card}(A_k) \leq 10^k.$$

Fro $k \in \mathbb{Z}_{>0}$ let Let

$$z_k = \max(A_k), \qquad \text{and let} \qquad z = (z_1, z_2, \ldots).$$

Check that $z = (z_1, z_2, \ldots)$ is a Cauchy sequence in $\mathbb{Q}_{\geq 0}$ and then check the defining conditions for $\sup(A)$ to complete the proof that the element of $\mathbb{R}_{\geq 0}$ given by the Cauchy sequence $z = (z_1, z_2, \ldots)$ is $\sup(A)$. □

17.3.3 Convergence and continuity in $\mathbb{R}_{\geq 0}$

Proposition 17.18.

(a) If $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, \ldots)$ converges to $\sup\{a_1, a_2, \ldots\}$.

(b) $\overline{\mathbb{Q}_{\geq 0}} = \mathbb{R}_{\geq 0}$.

Proof.

(a) Let $(a_1, a_2, \ldots)$ be a sequence in $\mathbb{R}$ such that $a_1 \leq a_2 \leq \cdots$ and there exists $b \in \mathbb{R}$ such that if $i \in \mathbb{Z}_{>0}$ then $a_i < b$.
By the least upper bound property (Proposition 17.9), since $A = \{a_1, a_2, \ldots\}$ is bounded then $\sup\{a_1, a_2, \ldots\}$ exists.
Let $c = \sup\{a_1, a_2, \ldots\}$.
To show: $\lim_{n\to\infty} a_n = c$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $|c - a_n| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $|c - a_n| < \epsilon$.
Using that $c - \epsilon$ is not an upper bound, let $\ell \in \mathbb{Z}_{>0}$ be such that $a_\ell > c - \epsilon$.
If $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \geq a_\ell$ and so $c - a_n \leq c - a_\ell < \epsilon$.

So $\lim_{n\to\infty} a_n = c$.
So $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, \ldots\}$.

(b) Let $x = z.d_1d_2\ldots \in \mathbb{R}_{\geq 0}$.
Let $x_k = z.d_1d_2\ldots d_k$ be the first k decimal places of x.
Then $(x_1, x_2, \ldots)$ is a sequence in $\mathbb{R}_{\geq 0}$ such that $\lim_{k\to\infty} x_k = x$.
So $\mathbb{R}_{\geq 0} \subseteq \overline{\mathbb{Q}_{\geq 0}}$.
Since $\overline{\mathbb{Q}_{\geq 0}}$ means closure of $\mathbb{Q}_{\geq 0}$ in $\mathbb{R}_{\geq 0}$ then $\overline{\mathbb{Q}_{\geq 0}} \subseteq \mathbb{R}_{\geq 0}$.
So $\overline{\mathbb{Q}_{\geq 0}} = \mathbb{R}_{\geq 0}$.

$\square$

Theorem 17.19. *Let $n \in \mathbb{Z}_{>0}$. The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, bijective, and satisfies*

$$\text{if } x, y \in \mathbb{R}_{\geq 0} \text{ and } x < y \quad \text{then} \quad x^n < y^n.$$

Furthermore, the inverse function $x^{\frac{1}{n}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.

Proof. (Sketch)
To show: (a) The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is monotone.
(b) The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is injective.
(c) The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is surjective.
(d) The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.
(e) The inverse function $x^{1/n} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ exists and is continuous.

(a) Assume $x, y \in \mathbb{R}_{\geq 0}$ and $x < y$.
Then there exists $z \in \mathbb{R}_{\geq 0}$ such that $x + z = y$.
Using the binomial theorem,

$$x^n < x^n + z^n < x^n + \Big(\sum_{j=1}^{n-1} \binom{n}{j} x^{n-j} y^j\Big) + y^n = (x+z)^n = y^n.$$

So the function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is monotone.

(b) Assume $x, y \in \mathbb{R}_{\geq 0}$ and $x \neq y$.
By (a), if $x < y$ then $x^n < y^n$ and $x^n \neq y^n$ and if $x > y$ then $x^n > y^n$ and $x^n \neq y^n$.
So the function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ then the function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is injective.

(c) To show: The function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is surjective.
To show: If $z \in \mathbb{R}_{\geq 0}$ then there exists $x \in \mathbb{R}_{\geq 0}$ such that $x^n = z$.
Assume $z \in \mathbb{R}_{\geq 0}$.
By the least upper bound property (Proposition 17.9), $z = \sup\{y \in \mathbb{R}_{\geq 0} \mid y^n < x\}$ exists in $\mathbb{R}_{\geq 0}$.
Then $z^n = x$.
So the function $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is surjective.

(d) To show: If $a \in \mathbb{R}_{\geq 0}$ then $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous at a.
Assume $a \in \mathbb{R}_{\geq 0}$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $y \in \mathbb{R}_{\geq 0}$ and $d(y, a) < \delta$ then $d(y^n, a^n) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.

To show: There exists $\delta \in \mathbb{E}$ such that if $d(y,a) < \delta$ then $d(y^n, a^n) < \epsilon$.
Let $\delta = \frac{1}{2^n a^{n-1}}\epsilon$.
Letting $d = d(a,y)$ then

$$d(y^n, a^n) = |y^n - a^n| = |(a+d)^n - a^n| = da^{n-1} \cdot \Big(\sum_{j=1}^{n} \frac{d^{j-1}}{a^{j-1}}\binom{n}{j}\Big)$$
$$< da^{n-1}\Big(\sum_{j=1}^{n}\binom{n}{j}\Big) = da^{n-1}(2^n - 1) < \delta 2^n a^{n-1} = \epsilon.$$

(What is at the core of this is that the distance $d(y^n, a^n)$ is related to the distance $d(y,a)$ by

$$d(y,a)a^{n-1}n < d(y^n, a^n) < d(y,a)a^{n-1}(2^n - 1).$$

So $x^n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous at a.
So $x^n \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.

(e) To show: If $b \in \mathbb{R}_{\geq 0}$ then $x^{\frac{1}{n}} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous at b.
Assume $b \in \mathbb{R}_{\geq 0}$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $z \in \mathbb{R}_{\geq 0}$ and $d(z,b) < \delta$ then $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
Let $\delta = na^{n-1}\epsilon^n$.
To show: If $z \in \mathbb{R}_{\geq 0}$ and $d(z,b) < \delta$ then $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Assume $z \in \mathbb{R}_{\geq 0}$ and $d(z,b) < \delta$.
To show: $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Let $a = b^{1/n}$ and $y = z^{1/n}$. Then

$$d(z^{1/n}, b^{1/n}) = d(y,a) < \frac{1}{na^{n-1}} d(y^n, a^n) = \frac{1}{na^{n-1}} d(z,b) < \frac{1}{na^{n-1}}\delta = \epsilon.$$

Since

$$d(y^n, a^n) = |y^n - a^n| = |(a+d)^n - a^n| = da^{n-1} \cdot \Big(\sum_{j=1}^{n} \frac{d^{j-1}}{a^{j-1}}\binom{n}{j}\Big) > da^{n-1}\binom{n}{1} = da^{n-1}n.$$

□

17.3.4 Topological properties of $\mathbb{R}_{\geq 0}$

Proposition 17.20. *(Topological properties of $\mathbb{R}_{\geq 0}$)*

(a) $\mathbb{R}_{\geq 0}$ is a Hausdorff topological space.

(b) $\mathbb{R}_{\geq 0}$ is a complete metric space.

(c) $\mathbb{R}_{\geq 0}$ is locally compact.

(d) $\mathbb{R}_{\geq 0}$ is not compact.

Proof. (Sketch)

(a) To show: If $x, y \in \mathbb{R}_{\geq 0}$ and $x \neq y$ then there exist open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Assume $x, y \in \mathbb{R}_{\geq 0}$ and $x \neq y$.
Let $\epsilon = \frac{1}{2}d(x,y)$ and let

$$U = \mathbb{R}_{(x-\epsilon,x+\epsilon)} \qquad \text{and} \qquad V = \mathbb{R}_{(y-\epsilon,y+\epsilon)}.$$

Then $x = x + 0 \in U$ and $y = y + 0 \in V$ and $U \cap V = \emptyset$.
So $\mathbb{R}_{\geq 0}$ is a Hausdorff topological space.

(b) To show: If $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$ then $(x_1, x_2, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.
To show: If $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$ then there exists $y \in \mathbb{R}_{\geq 0}$ such that $y = \lim_{n\to\infty} x_n$.
Let $(x_1, x_2, \ldots)$ be a Cauchy sequence in $\mathbb{R}_{\geq 0}$.

$$\begin{aligned} x_1 &= z_1.d_{11}d_{12}d_{13}\ldots, \\ x_2 &= z_2.d_{21}d_{22}d_{23}\ldots, \\ x_3 &= z_3.d_{31}d_{32}d_{33}\ldots, \\ &\vdots \end{aligned}$$

To show: There exists $y \in \mathbb{R}_{\geq 0}$ such that $y = \lim_{n\to\infty} x_n$.
For $k \in \mathbb{Z}_{\geq 0}$ let ℓ_k be such that if $m, n \in \mathbb{Z}_{\geq \ell_k}$ then $d(x_m, x_n) \leq 10^{-k}$.
Let $y = z.d_1d_2d_3\cdots$, where

$$z = z_{\ell_0}, \quad d_1 = d_{\ell_1 1}, \quad d_2 = d_{\ell_2 2}, \quad \ldots$$

To show: If $k \in \mathbb{Z}_{\geq 0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq N}$ then $d(x_m, y) < 10^{-k}$.
Assume $k \in \mathbb{Z}_{>0}$.
Let $N = \ell_{k+1}$.
To show: If $m \in \mathbb{Z}_{\geq \ell_{k+1}}$ then $d(x_m, y) < 10^{-k}$.
Assume $m \in \mathbb{Z}_{\geq \ell_{k+1}}$.
Then

$$d(x_m, y) \leq d(x_m, x_{\ell_{k+1}}) + d(x_{\ell_{k+1}}, y) < 10^{-(k+1)} + 10^{-(k+1)} < 10^{-k}.$$

So $\lim_{k\in\infty} x_k = y$.
So Cauchy sequences in $\mathbb{R}_{\geq 0}$ converge.
So $\mathbb{R}_{\geq 0}$ is complete.
This proof is conceptual and easy but there is a little bit of fuzziness in this proof caused by the fact that the decimal expansion of an element of $\mathbb{R}_{\geq 0}$ is not uniquely determined, for example $0.999\ldots = 1.000\ldots$. To remove this fuzziness use equivalence classes of Cauchy sequences in $\widehat{\mathbb{Q}}_{\geq 0}$ as in the proof that the completion of a metric space is complete.

(c) To show: $\mathbb{R}_{\geq 0}$ is locally compact.
To show: (ca) $\mathbb{R}_{\geq 0}$ is Hausdorff.
(cb) If $x \in \mathbb{R}_{\geq 0}$ then there exists a neighborhood N of x such that N is cover compact.

(ca) By part (b), $\mathbb{R}_{\geq 0}$ is Hausdorff.
(cb) To show: If $x \in \mathbb{R}_{\geq 0}$ then there exists a neighborhood N of x such that N is cover compact.
Assume $x \in \mathbb{R}_{\geq 0}$.

Let $N = \overline{B_1(x)} = \{y \in \mathbb{R}_{\ge 0} \mid |y - x| \le 1\}$.
Since $N \supseteq B_1(x)$ and $x \in B_1(x)$ then N is a neighborhood of x.
Since $N \subseteq B_2(x)$ then N is bounded.
Since N is closed and bounded then N is cover compact.

So $\mathbb{R}_{\ge 0}$ is locally compact.

(d) The sequence $(1, 2, 3, 4, \ldots)$ is a sequence in $\mathbb{R}_{\ge 0}$ that does not have a cluster point.
So $\mathbb{R}_{\ge 0}$ is not compact.

□

An *interval* in $\mathbb{R}_{\ge 0}$ is a set $A \subseteq \mathbb{R}_{\ge 0}$ such that

$$\text{if } x, y \in A \text{ and } z \in \mathbb{R}_{\ge 0} \text{ and } x < z < y \text{ then } z \in A.$$

Theorem 17.21. *Let $A \subseteq \mathbb{R}_{\ge 0}$.*

(a) A is connected if and only if A is an interval.

(b) A is compact if and only if A is closed and bounded.

Proof.

(a) $\Rightarrow$: Assume E is not an interval.
Let $x, y \in E$ and $z \in E^c$ with $x < z < y$.
Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$.
Then A and B are open sets of J and, since $x \in A$ and $y \in B$ then

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap B = \emptyset, \quad \text{and} \quad A \cup B = E.$$

So E is not connected.

(a) $\Leftarrow$: Assume E is an interval.
To show: E is connected.
Let $A \subseteq E$ and $B \subseteq E$ be open subsets of E such that

$$A \neq \emptyset, \quad B \neq \emptyset \quad \text{and} \quad A \cup B = E.$$

To show: $A \cap B \neq \emptyset$.
There exists $z \in A \cap B$.
Let $x_1, y_1 \in J$ with $x_1 \in A$ and $y_1 \in B$.
Switching A and B if necessary assume that $x_1 < y_1$.
Construct sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ by

$$x_{i+1} = \frac{x_i + y_i}{2} \quad \text{and} \quad y_{i+1} = y_i, \qquad \text{if } \frac{x_i + y_i}{2} \in A,$$

$$x_{i+1} = x_i \quad \text{and} \quad y_{i+1} = \frac{x_i + y_i}{2}, \qquad \text{if } \frac{x_i + y_i}{2} \in B.$$

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By induction, $x_i \in E$ and $y_i \in E$, and since E is an interval, $\frac{1}{2}(x_i + y_i) \in E$ so that

$$x_{i+1} \in E \quad \text{and} \quad y_{i+1} \in E.$$

Also

$$x_{i+1} \in A, \quad y_{i+1} \in B, \qquad x_i \le x_{i+1} < y_{i+1} \le y_i,$$

and

$$|x_{i+1} - y_{i+1}| \le \tfrac{1}{2}|x_i - y_i|, \qquad \text{so that} \qquad |x_{i+1} - y_{i+1}| \le \frac{1}{2^i}|x_1 - y_1|.$$

Theorem 17.10(a) says that increasing bounded sequences converge, and since the sequence $x_1, x_2, \ldots$ is increasing and bounded by y_1
then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.
Theorem 17.10(a) says that decreasing bounded sequences converge, and since the sequence $y_1, y_2, \ldots$ is decreasing and bounded by x_1
then $\lim_{n\to\infty} y_n$ exists in $\mathbb{R}$.
Since $\lim_{n\to\infty} |x_n - y_n| = 0$ then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.
Let

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Since $x_1 \le x_2 \le \cdots \le x_n < y_n \le y_{n-1} \le \cdots \le y_1$ for $n \in \mathbb{Z}_{>0}$ then

$$x_1 < z < y_1.$$

Since E is an interval, $z \in E$.
By the characterization of closure in metric spaces via limits (Theorem 13.6),

$$z = \lim_{n\to\infty} x_n \in \overline{A} \qquad \text{and} \qquad z = \lim_{n\to\infty} y_n \in \overline{B}.$$

Since $\overline{A} = A$ and $\overline{B} = B$ then $z \in A \cap B$.
So $A \cap B \ne \emptyset$.
So E is connected.

(b) By Theorem 4.1, E is compact if E is Cauchy compact and bounded, so
To show: (ba) If $E \subseteq \mathbb{R}$ is bounded then E is ball compact.
(bb) If $E \subseteq \mathbb{R}$ is closed then E is Cauchy compact.

(ba) Assume $E \subseteq \mathbb{R}$ is bounded.
To show: E is ball compact.
Since E is bounded there exists $x \in \mathbb{R}$ and $M \in \mathbb{R}_{>0}$ such that $E \subseteq (x - M, x + M)$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$ such that $E \subseteq B_\epsilon(x_1) \cup \cdots B_\epsilon(x_\ell)$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$ such that $E \subseteq B_\epsilon(x_1) \cup \cdots B_\epsilon(x_\ell)$.
Let $\ell \in \mathbb{Z}_{>0}$ such that $\ell \cdot \frac{\epsilon}{2} > 2M$. Let

$$x_1 = x - M, \quad x_2 = x_1 + \tfrac{\epsilon}{2}, \quad x_3 = x_2 + \tfrac{\epsilon}{2}, \ldots, x_\ell = x_1 + \ell\tfrac{\epsilon}{2}.$$

Then

$$\begin{aligned} E &\subseteq (x - M, x + M) \\ &\subseteq (x_1 - \tfrac{\epsilon}{2}, x_1 + \tfrac{\epsilon}{2}) \cup (x_2 - \tfrac{\epsilon}{2}, x_2 + \tfrac{\epsilon}{2}) \cup \cdots (x_\ell - \tfrac{\epsilon}{2}, x_\ell + \tfrac{\epsilon}{2}). \end{aligned}$$

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So E is ball compact.

(bb) Assume E is closed.
To show: E is Cauchy compact.
To show: E is complete.
To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in E then $(a_1, a_2, \ldots)$ converges in E.
Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in E.
Then $(a_1, a_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}$ is complete then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$.
To show: $\lim_{n\to\infty} a_n$ is an element of E.
Since E is closed,

$$E = \overline{E} = \big\{z \in \mathbb{R} \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } E \text{ with } z = \lim_{n\to\infty} a_n\big\}.$$

So $\lim_{n\to\infty} a_n \in \overline{E} = E$.
So $(a_1, a_2, \ldots)$ converges in E.
So E is complete.

So E is ball compact and Cauchy compact in the metric space $\mathbb{R}$.

So E is compact.

□

17.3.5 Notes and References

AN IMPORTANT QUESTION IS HOW TO COMPUTE EXPLICITLY THE DECIMAL EXPANSIONS OF $a + b$, ab and a^{-1}. NOTE THAT multiplication is not uniformly continuous. ALSO WE NEED TO VERIFY THAT THESE OPERATIONS ARE WELL DEFINED.

To construct x^{-1} compute 1 divided by x by long division. Alternatively, multiply x by 10^{-k} to get a number y less than 1 and let z be such that $y + z = 1$. Then

$$\frac{1}{x} = \frac{1}{10^k y} = 10^{-k}\frac{1}{y} = 10^{-k}\frac{1}{1-z} = 10^{-k}(1 + z + z^2 + \ldots).$$