3+3+10



Question 1:

(a) Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.

The closure of $A$ is $\overline{A} \subseteq X$ such that

(a) $\overline{A}$ is closed in $X$ and $\overline{A} \supseteq A$,

(b) If $C \subseteq X$ is closed in $X$ and $C \supseteq A$ then $C \supseteq \overline{A}$.

(b) Theorem Let $(X, d)$ be a metric space. Let $A \subseteq X$. Then

$$\overline{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence} \\ (a_1, a_2, \dots) \text{ in } A \text{ such that} \\ \lim_{n \to \infty} a_n = z \end{array} \right\}$$

(c) Proof: Let $R = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence} \\ (a_1, a_2, \dots) \text{ in } A \text{ such that} \\ \lim_{n \to \infty} a_n = z \end{array} \right\}$

To show: $R = \overline{A}$.

To show: (ca) $R \subseteq \overline{A}$

(cb) $\overline{A} \subseteq R$.

(ca) To show: If $z \in R$ then $z \in \overline{A}$.

Assume $z \in R$.

To show: $z \in \bar{A}$

To show: $z$ is a close point to $A$.

To show: If $N \in \mathcal{N}(z)$ then $N \cap A \neq \phi$.

Assume $N \in \mathcal{N}(z)$.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(z) \subseteq N$.

To show: $B_\varepsilon(z) \cap A \neq \phi$.

Since $z \in R$ there exists a sequence $(a_1, a_2, \ldots)$ in $A$ such that $\lim_{n \to \infty} a_n = z$.

So there exists $l \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq l}$ then $d(a_n, z) < \varepsilon$.

So if $n \in \mathbb{Z}_{\geq l}$ then $a_n \in B_\varepsilon(z)$.

So $B_\varepsilon(z) \cap A \neq \phi$.

So $z$ is a close point to $A$.

So $z \in \bar{A}$.

(cb) To show: $\bar{A} \subseteq R$.

To show: If $z \in \bar{A}$ then $z \in R$.

Assume $z \in \bar{A}$.

Let $(a_1, a_2, \ldots)$ be in $A$ such that

$a_1 \in B_{0.1}(z) \cap A$, $a_2 \in B_{0.01}(z) \cap A$, ....

Q1 ③

These exist since $z$ is a close point to $A$

To show: $\lim\limits_{n\to\infty} a_n = z$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ then $d(a_n, z) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\ell \in \mathbb{Z}_{>0}$ such that $\frac{1}{10^\ell} < \varepsilon$.

~~Then~~ Assume $n \in \mathbb{Z}_{\geq \ell}$. Then $a_n \in B_{10^{-n}}(z)$

and

$$d(a_n, z) < \frac{1}{10^n} \leq \frac{1}{10^\ell} < \varepsilon.$$

So $\lim\limits_{n\to\infty} a_n = z$. //

5+5+10



## Question 2

(a) A normed ~~lt~~ vector space is a ~~lt~~ vector space $X$ ~~such that~~ with a function $\| \ \|: X \to \mathbb{R}_{\geq 0}$ such that

(a) If $c \in \mathbb{K}$ and $x \in X$ then $\|cx\| = |c| \cdot \|x\|$,

(b) If $x, y \in X$ then $\|x+y\| \leq \|x\| + \|y\|$.

(c) If $x \in X$ and $\|x\| = 0$ then $x = 0$.

(b) Let $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$ be normed vector spaces. The space of bounded linear operators from $V$ to $W$ is

$$B(V, W) = \left\{ T: V \to W \;\middle|\; \begin{array}{c} T \text{ is a linear transformation} \\ \|T\| < \infty \end{array} \right\}$$

with

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \;\middle|\; v \in V \text{ and } v \neq 0 \right\}.$$

(c) To show: $B(V, W)$ is a normed vector space.

First note that addition and scalar multiplication in $B(V, W)$ are defined by

Q2 ②

$(T_1+T_2): V \to W$ and $(cT): V \to W$ with

$(T_1+T_2)(v) = T_1(v) + T_2(v)$, and

$(cT)(v) = c \cdot T(v)$

for $T_1, T_2, T \in B(V,W)$ and $c \in \mathbb{K}$.

To show: (a) $B(V,W)$ is a vector space.

(b) $\| \ \|$ defines a norm on $B(V,W)$.

(b) To show: (ba) If $T_1, T_2 \in B(V,W)$ then

$\|T_1+T_2\| \le \|T_1\| + \|T_2\|$

(bb) If $T \in B(V,W)$ and $c \in \mathbb{K}$ then

$\|cT\| = |c| \cdot \|T\|$

(bc) If $T \in B(V,W)$ and $\|T\| = 0$ then $T = 0$.

(bc) Assume $T \in B(V,W)$ and $\|T\| = 0$.

Then $\sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0 \right\} = 0$.

So $\|Tv\| = 0$ for $v \in V$ with $v \neq 0$.

So $Tv = 0$ for $v \in V$ with $v \neq 0$.

Since $T$ is a linear transformation $T(0) = 0$.

So $Tv = 0$ for $v \in V$.

So $T = 0$.

Q2

(3)

(6a) Assume $T_1, T_2 \in B(V,W)$.

Let $v \in V$. ~~with~~ Then, by the triangle inequality,

$$\|(T_1+T_2)(v)\|_V = \|T_1 v + T_2 v\|_V \leq \|T_1 v\|_V + \|T_2 v\|_V$$

By the definition of $\|T_1\|$ and $\|T_2\|$ then

$$\|(T_1+T_2)(v)\|_W \leq \|T_1 v\|_W + \|T_2 v\|_W \leq \|T_1\| \cdot \|v\|_V + \|T_2\| \cdot \|v\|_V$$

$$= (\|T_1\| + \|T_2\|)\|v\|_V.$$

So $\dfrac{\|(T_1+T_2)(v)\|_W}{\|v\|_V} \leq \|T_1\| + \|T_2\|$ for $v \neq 0$.

So $\|T_1+T_2\| = \sup\left\{ \dfrac{\|(T_1+T_2)(v)\|_W}{\|v\|_V} \,\middle|\, v \in V \text{ and } v \neq 0 \right\}$

$$\leq \|T_1\| + \|T_2\|.$$

6+6+6+6



Question 3 Let $(X,d)$ be a metric space.

(a) Let $(x_1, x_2, \ldots)$ be a sequence in $X$.

A limit point of $(x_1, x_2, \ldots)$ is $z \in X$ such that if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, z) < \varepsilon$.

A cluster point of $(x_1, x_2, \ldots)$ is $z \in X$ such that there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $z$ is a limit point of $(x_{n_1}, x_{n_2}, \ldots)$.

(b) A Cauchy sequence in $X$ is a sequence $(x_1, x_2, \ldots)$ in $X$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_r, x_s) < \varepsilon$.

A convergent sequence in $X$ is a sequence $(x_1, x_2, \ldots)$ in $X$ such that

there exists $z \in X$ such that $z$ is a limit point of $(x_1, x_2, \ldots)$.

(c) Let $(X,d)$ be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in ~~X~~. If $z \in X$ is a limit point of $(x_1, x_2, \ldots)$ then $z$ is a cluster point of $(x_1, x_2, \ldots)$.

Proof Assume $z \in X$ is a limit point of $(x_1, x_2, \ldots)$.

To show: $z$ is a cluster point of $(x_1, x_2, \ldots)$

To show: There exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $z$ is a limit point of $(x_{n_1}, x_{n_2}, \ldots)$.

Let $x_{n_1} = x_1$, $x_{n_2} = x_2$, $\ldots$ .

Then $z$ is a limit point of $(x_{n_1}, x_{n_2}, \ldots) = (x_1, x_2, \ldots)$.

So $z$ is a cluster point of $(x_1, x_2, \ldots)$.

(d) Let $(X,d)$ be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in $X$. If $(x_1, x_2, \ldots)$ is convergent then $(x_1, x_2, \ldots)$ is Cauchy.

Proof Assume $(x_1, x_2, \ldots)$ is convergent.

So there exists $z \in X$ such that $\lim_{n \to \infty} x_n = z$.

To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_r, x_s) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then

$$d(x_n, z) < \frac{\varepsilon}{10}.$$

To show: if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_r, x_s) < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{\geq \ell}$.

To show: $d(x_r, x_s) < \varepsilon$.

$$d(x_r, x_s) \leq d(x_r, z) + d(z, x_s)$$

$$< \frac{\varepsilon}{10} + \frac{\varepsilon}{10} = \frac{1}{5}\varepsilon < \varepsilon.$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in $X$.

6+6+6



## Question 4

(a) A ___topological space___ is a set $X$ with a collection $\mathcal{T}$ of subsets of $X$ such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $(U_1 \cap U_2 \cap \cdots \cap U_\ell) \in \mathcal{T}$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. ~~Let~~ A ___continuous function from $X$ to $Y$___ is a function $f: X \to Y$ such that

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$,

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

(b) A ___uniform space___ is a set $X$ with a collection $\mathcal{E}$ of subsets of $X \times X$ such that

(a) If $E \in \mathcal{E}$ then $E \supseteq \Delta(X)$, where $\Delta(X) = \{(x,x) \mid x \in X\}$.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces.

A <u>uniformly continuous function</u> from $X$ to $Y$ is a function $f: X \to Y$ such that

if $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.

(c) Assume $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are uniform spaces.

Let $\mathcal{T}_X$ be the uniform space topology on $X$ and

$\mathcal{T}_Y$ the uniform space topology on $Y$.

Let $f: X \to Y$ be a uniformly continuous function.

To show: $f: X \to Y$ is continuous.

To show: If $a \in X$ then $f$ is continuous at $a$.

Assume $a \in X$.

To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

Assume $N \in \mathcal{N}(f(a))$.

By the definition of uniform space topology there exists $E \in \mathcal{E}_Y$ such that $B_E(f(a)) \subseteq N$.

To show: There exists $D \in \mathcal{E}_X$ such that $B_D(a) \subseteq f^{-1}(N)$.

Let $D = (f \times f)^{-1}(E)$.

Since $f$ is uniformly continuous $D \in \mathcal{E}_X$.

To show $B_D(a) \subseteq f^{-1}(N)$.

Q4

③

To show: If $x \in B_D(a)$ then $x \in f^{-1}(N)$.

Assume $x \in B_D(a)$

So $(x, a) \in D = (f \times f)^{-1}(E)$.

So $(f \times f)(x, a) \in E$.

So $(f(x), f(a)) \in E$.

So $f(x) \in B_E(f(a)) \subseteq N$.

So $x \in f^{-1}(N)$.

So $f$ is continuous at $a$.

So $f$ is continuous. $\Delta$.