5+5+10

Question 5

(a) Let (X, d_X) and (Y, d_Y) be metric spaces.

Let $\mathcal{T}_X$ be the metric space topology on X and $\mathcal{T}_Y$ the metric space topology on Y.

The metric spaces (X, d_X) and (Y, d_Y) are topologically equivalent if $\mathcal{T}_X = \mathcal{T}_Y$.

(b) The standard metric on $\mathbb{R}^2$ is $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$

(c) Let $d_2((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|$.

Let $\mathcal{T}_1$ be the metric space topology on $(\mathbb{R}^2, d)$

Let $\mathcal{T}_2$ be the metric space topology on $(\mathbb{R}^2, d_2)$.

To show: $\mathcal{T}_1 = \mathcal{T}_2$.

Let $B_\varepsilon^d(x) = \{ y \in \mathbb{R}^2 \mid d(x, y) < \varepsilon \}$ and

$B_\varepsilon^{d_2}(x) = \{ y \in \mathbb{R}^2 \mid d_2(x, y) < \varepsilon \}$

To show: (ca) If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta^{d_2}(x) \subseteq B_\varepsilon^{d}(x)$.

(cb) If $\delta \in \mathbb{R}_{>0}$ then there exists $\gamma \in \mathbb{R}_{>0}$ such that $B_\gamma^{d}(x) \subseteq B_\delta^{d_2}(x)$.

(ca) Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\delta = \varepsilon$

To show: $B_\delta^{d_2}(x) \subseteq B_\varepsilon^{d}(x)$.

To show: If $y \in B_\delta^{d_2}(x)$ then $y \in B_\varepsilon^{d}(x)$.

Assume $y = (y_1, y_2) \in B_\delta^{d}(x)$.

So ~~$(y_1-x_1)^2 + (y_2-x_2)^2 < \delta^2$~~ $|y_1 - x_1| + |y_2 - x_2| < \delta$.

To show: $(y_1-x_1)^2 + (y_2-x_2)^2 < \varepsilon^2$.

$$(y_1-x_1)^2 + (y_2-x_2)^2 \le (|y_1-x_1| + |y_2-x_2|)^2 < \delta^2 = \varepsilon^2$$

(cb) Assume $\delta \in \mathbb{R}_{>0}$.

Let $\gamma =$

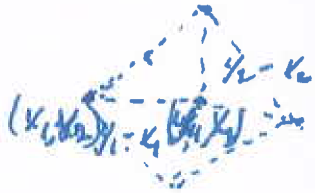
To show: $B_\gamma^{d}(x) \subseteq B_\delta^{d_2}(x)$

To show: If $y \in B_\gamma^{d}(x)$ then $y \in B_\delta^{d_2}(x)$.

Assume $y \in B_\gamma^{d}(x)$.

Then, with $y=(y_1,y_2)$ and $x=(x_1,x_2)$,

$$(y_1-x_1)^2+(y_2-x_2)^2<\gamma^2.$$



To show: $y\in B^{d_2}_{\delta}(x)$.

To show: $|y_1-x_1|+|y_2-x_2|<\delta$.

$$|y_1-x_1|+|y_2-x_2| = d((x_1,x_2),(y_1,x_2))+d((y_1,x_2),(y_1,y_2))$$

By the parallelogram law,

$$\|(y_1,y_2)-(x_1,x_2)\|_2+\|(y_1,y_2)+(x_1,x_2)\|_2$$

$$\|(x_1-x_2,\ y_1-y_2)\|_2+\|(x_1+x_2,\ y_1+y_2)\|_2 = \|2(x_1,y_1)\|_2+2\|(x_2,y_2)\|_2$$

So

$$\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\le 2\cdot 2\,|y_1-x_1|+|y_2-x_2| < 4\delta=\gamma.$$

So $\tau_1=\tau_2$

So $(\mathbb{R}^2,d)$ and $(\mathbb{R}^2,d_2)$ are topologically equivalent.

6+6

Question 6

(a) Theorem Let $(H, \langle\ \rangle)$ be a Hilbert space with a countable dense set. ~~Then~~ Let $T: H \to H$ be a bounded self adjoint compact operator. Then there exists an orthonormal basis of H consisting of eigenvectors of H.

(b) If H is finite dimensional then $H = \mathbb{C}^n$ with
$$\langle (x_1, x_2, \dots x_n), (y_1, y_2, \dots, y_n) \rangle = x_1\bar{y}_1 + \dots + x_n\bar{y}_n.$$

Say $H = \mathbb{C}^3$. Then $T: H \to H$ can be represented as a 3×3 matrix A.

All 3×3 matrices correspond to bounded linear transformations $\mathbb{C}^3 \to \mathbb{C}^3$, and all 3×3 matrices correspond to compact linear operators $\mathbb{C}^3 \to \mathbb{C}^3$.

If $T: \mathbb{C}^3 \to \mathbb{C}^3$ is self adjoint, then
$$A = \bar{A}^t \quad \text{(with respect to the standard basis)}$$

Q6

2

So $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ represents a bounded compact linear operator

There exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_3$ ~~such that~~ and $g \in GL_3(\mathbb{C})$ such that

$$g^{-1} A g = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$b_1 = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \frac{1}{\sqrt{1^2+2^2+3^2}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{b_1}{\|b_1\|}$$

$$b_2 = \frac{1}{\sqrt{14}} A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \\ 31 \end{pmatrix} \qquad \frac{1}{\sqrt{(14)^2+(25)^2+(31)^2}} \frac{1}{\sqrt{14}} \begin{pmatrix} 14 \\ 25 \\ 31 \end{pmatrix} = \frac{b_2}{\|b_2\|}.$$

Should already be an approximation of an eigenvector.

7+5+7+7

①

Question 7 Let (X, d_X) and (Y, d_Y) be metric spaces

(a) Let $f_1: X \to Y$, $f_2: X \to Y$, ... be functions.

Assume $f: X \to Y$ is a function and $(f_1, f_2, \ldots)$ converges uniformly to f.

To show: $(f_1, f_2, \ldots)$ converges pointwise to f.

To show: If $x \in X$ then $\lim_{n\to\infty} f_n(x) = f(x)$.

Assume $x \in X$.

To show: $\lim_{n\to\infty} f_n(x) = f(x)$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d_Y(f_n(x), f(x)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $(f_1, f_2, \ldots)$ converges uniformly to f then

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0, \quad \text{where}$$

$$d_\infty(f_n, f) = \sup\{ d_Y(f_n(p), f(p)) \mid p \in X\}.$$

So there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d_\infty(f_n, f) < \varepsilon$.

Q7 ②

To show: $d_Y(f_n(x), f(x)) < \varepsilon$.

$$d_Y(f_n(x), f(x)) \le \sup\{ d_Y(f_n(p), f(p)) \mid p \in X\}$$
$$= d_\infty(f_n, f) < \varepsilon.$$

So $d_Y(f_n(x), f(x)) < \varepsilon$.

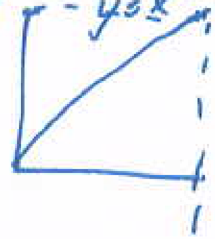
So $(f_1, f_2, \ldots)$ converges pointwise to f.

(b) Let $f_n : \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ and $f : \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$
$x \longmapsto x^n$

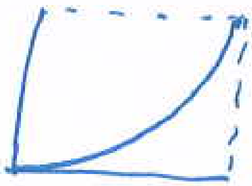
given by $f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1 \end{cases}$

$y = f_1(x) = x$

1 - - $y = x = f_1(x)$



$y = f_2(x) = x^2$

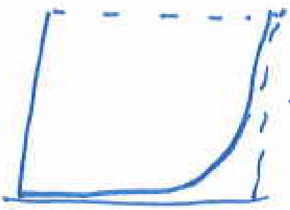


$y = x^2 = f_2(x)$.

$y = f_3(x) = x^3$

$y = x^3 = f_3(x)$

$y = f_4(x) = x^4$



$y = x^4 = f_4(x)$.

Q7

(3)

$y = f(x)$  $y = f(x).$

To show: (ba) If $x \in \mathbb{R}_{[0,1]}$ then $\lim_{n\to\infty} f_n(x) = f(x)$.

(bb) $(f_1, f_2, \ldots)$ does not converge uniformly to f.

(ba) Assume $x \in \mathbb{R}_{[0,1]}$

Case 1: $x = 1$. Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(1)$

$= \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1 = f(1) = f(x)$.

Case 2: $x < 1$. To show: $\lim_{n\to\infty} x^n = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x^n < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $k \in \mathbb{Z}_{>0}$ such that $x < 1 - \frac{1}{10^k} = \frac{10^k - 1}{10^k}$.

Let $M \in \mathbb{Z}_{>0}$ such that $\frac{1}{10^M} < \varepsilon$.

We want, if $n \in \mathbb{Z}_{\geq \ell}$.

$$x^n < x^\ell < \left(\frac{10^k - 1}{10^k}\right)^\ell < \frac{10^{k\ell} - \ell 10^k}{10^{k\ell}} = 1 - \frac{\ell}{10^{k(\ell-1)}} < \frac{1}{10^M} < \varepsilon$$

Q7 ④

and this happens when

$$10^{M}\frac{\ell 10^{M}}{10^{k(\ell-k)}}<1, \text{ i.e. } 10^{M}<\frac{1}{\frac{10^{k\ell}-10^{k}\ell}{10^{k\ell}}}$$

$$\text{so } 10^{M}<\frac{10^{k\ell}}{10^{k\ell}-10^{k}\ell}=\frac{10^{\ell k}-10^{k}\ell+10^{k}\ell}{10^{\ell k}-\ell 10^{k}}=1+\frac{10^{k\ell}}{10^{k\ell}-\ell 10^{k}}.$$

(bb) To show: $\lim_{n\to\infty} d_\infty(f_n, f) \neq 0$.

To show: There exists $\ell\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{>\ell}$ then $d_\infty(f_n,f)>\frac{1}{2}$.

To show: If $n\in\mathbb{Z}_{>0}$ then $d_\infty(f_n,f)=1$.

Assume $n\in\mathbb{Z}_{>0}$

To show: If $\varepsilon\in\mathbb{R}_{>0}$ there exists $x\in\mathbb{R}_{[0,1]}$ such that $d_y(x^n, 0)>1-\varepsilon$

Assume $\varepsilon\in\mathbb{R}_{>0}$.

Let $k\in\mathbb{Z}_{>0}$ such that $\frac{1}{10^k}<\varepsilon$.

To show: there exists $x\in\mathbb{R}_{[0,1]}$ such that $x^n>1-\frac{1}{10^k}$

Let $r\in\mathbb{Z}_{>0}$ such that $10^r>n10^k$ and let $x=1-\frac{1}{10^r}$.

Then $x^n = \left(1-\frac{1}{10^r}\right)^n > 1-n\frac{1}{10^r} > 1-\frac{1}{10^k}$

since $\frac{1}{10^r}<\frac{1}{n10^k}$. //.

$6+6+6+6$

1

Question 8 Let $(a_1, a_2, \ldots)$ be a sequence in $\mathbb{C}$ with $\sup\{|a_1|, |a_2|, \ldots\} < \infty$. Let

$T: \ell^2 \to \ell^2$ given by $T(b_1, b_2, \ldots) = (0, a_1 b_1, a_2 b_2, \ldots)$

Let $C = \sup\{|a_1|, |a_2|, \ldots\}$.

(a) To show: ~~(aa)~~ $\|T\| = C$. (this will give that T is a bounded operator)

To show: (aa) $\|T\| \le C$

(ab) $\|T\| \ge C$.

(ab) Let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ with 1 in the ith spot.

Then

$$\|Te_i\|_2 = \|a_i e_{i+1}\|_2 = |a_i| \cdot \|e_{i+1}\|_2 = |a_i| \cdot \|e_i\|_2.$$

So $\|T\| \ge |a_i|$ for $i \in \mathbb{Z}_{>0}$

So $\|T\| \ge \sup\{|a_1|, |a_2|, \ldots\} = C$.

(aa) Let $b = (b_1, b_2, \ldots) \in \ell^2$ so that $\|b\|_2 < \infty$.

Then

$$\|Tb\|_2 = \|(0, a_1 b_1, a_2 b_2, \ldots)\|_2$$

$$= \left(\sum_{i=1}^{\infty} |a_i b_i|^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |a_i|^2 |b_i|^2\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{i=1}^{\infty} C^2 |b_i|^2\right)^{\frac{1}{2}} = \left(C^2 \cdot \sum_{i=1}^{\infty} |b_i|^2\right)^{\frac{1}{2}} = C\|b\|_2.$$

So $\|T\| \le C$

(b) The adjoint operator will satisfy

$$\langle T^*(c_1,c_2,\dots), e_j\rangle = \langle (c_1,c_2,\dots), Te_j\rangle$$

$$= \langle (c_1,c_2,\dots), a_j e_{j+1}\rangle = \bar{a}_j c_{j+1}.$$

So $T^*(c_1,c_2,\dots) = (\bar{a}_1 c_2, \bar{a}_2 c_3, \dots)$.

(c) $T^*T(1,0,0,\dots) = T^*(0,a_1,0,0,\dots)$

$$= (\bar{a}_1 a_1, 0, 0, \dots)$$

and

$$TT^*(1,0,0,\dots) = T(0,0,0,\dots)$$

$$= (0,0,0,\dots)$$

So $T^*T \ne TT^*$ when $a_1 \ne 0$.

Similarly, if $j \in \mathbb{Z}_{>0}$ such that $a_j \ne 0$ then

$T^*Te_j = \bar{a}_j a_j e_j$ and

$TT^*e_j = T(\bar{a}_j e_{j-1}) = \bar{a}_j a_{j-1} e_j = a_j \cdot 0\, e_j = 0$

so that $T^*T \ne TT^*$.

Q8

(d) If $T^*(c_1, c_2, \dots) = \lambda(c_1, c_2, \dots)$ then

$(\bar{a}_1 c_2, \bar{a}_2 c_3, \bar{a}_3 c_4, \dots) = (\lambda c_1, \lambda c_2, \dots)$ so that

$\bar{a}_1 c_2 = \lambda c_1$
$\bar{a}_2 c_3 = \lambda c_2$
$\vdots$

so that $c_2 = (\bar{a}_1)^{-1} \lambda c_1$
$c_3 = (\bar{a}_2)^{-1} \lambda c_2 = (\bar{a}_2 \bar{a}_1)^{-1} \lambda^2 c_2$
$\vdots$

when $\bar{a}_1 \neq 0, \bar{a}_2 \neq 0, \dots$.

Normalizing $c_1 = 1$, then,

(A) if $\bar{a}_i \neq 0$ then $(1, (\bar{a}_1)^{-1}\lambda, (\bar{a}_1 \bar{a}_2)^{-1}\lambda^2, \dots) = v$
is the unique (up to constant) eigenvector of eigenvalue λ.

(B) if $\bar{a}_k = 0$ and $\lambda \neq 0$ then $c_{k-1} = 0, c_{k-2} = 0, \dots, c_1 = 0$

and $(0, 0, \dots, 0, 1, (\bar{a}_k)^{-1}\lambda, (\overline{a_k a_{k-1}})^{-1}\lambda^2, \dots) = v$
is the unique (up to constant) eigenvector of eigenvalue λ.