8 Sequences and series: Review from Calculus 2

8.1 Sequences

Let Y be a set. A *sequence* $(y_1, y_2, y_3, \ldots)$ in Y is a function

$$\begin{matrix} \mathbb{Z}_{>0} & \longrightarrow & Y \\ n & \longmapsto & y_n \end{matrix}$$

Let Y be a set with a partial order $\le$ and let $(y_1, y_2, y_3, \ldots)$ be a sequence in Y.

- The sequence $(y_1, y_2, y_3, \ldots)$ is *increasing* if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \qquad \text{then} \quad y_i \le y_{i+1}.$$

- The sequence $(y_1, y_2, y_3, \ldots)$ is *decreasing* if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \qquad \text{then} \quad y_i \ge y_{i+1}.$$

- The sequence $(y_1, y_2, y_3, \ldots)$ is *monotone* if it is increasing or decreasing.

Let Y be a metric space and let $(y_1, y_2, y_3, \ldots)$ be a sequence in Y.

- The sequence $(y_1, y_2, y_3, \ldots)$ is *bounded* if the set $\{y_1, y_2, y_3, \ldots\}$ is bounded.
- The sequence $(y_1, y_2, y_3, \ldots)$ is *Cauchy* if $(y_1, y_2, \ldots)$ satisfies:

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then} \quad \text{there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } m, n \in \mathbb{Z}_{\ge N} \text{ then } d(y_m, y_n) < \varepsilon.$$

- Let $\ell \in Y$. The sequence $(y_1, y_2, y_3, \ldots)$ *converges to* ℓ if

$$\lim_{n\to\infty} y_n = \ell$$

  i.e., if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\ge N} \text{ then } d(y_n, \ell) < \varepsilon.$$

- The sequence $(y_1, y_2, \ldots)$ *converges in* Y if there exists $\ell \in Y$ such that $(y_1, y_2, \ldots)$ converges to ℓ.
- The sequence $(y_1, y_2, \ldots)$ *diverges in* Y if there does not exist $\ell \in Y$ such that $(y_1, y_2, \ldots)$ converges to ℓ.

Let $(y_1, y_2, y_3, \ldots)$ be a sequence in $\mathbb{R}$.

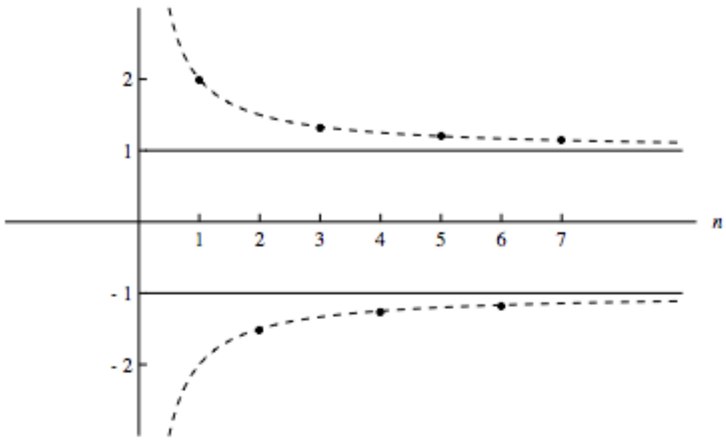
- The *supremum* of $(y_1, y_2, y_3, \ldots)$ is $\sup\{y_1, y_2, y_3, \ldots\}$.
- The *infimum* of $(y_1, y_2, y_3, \ldots)$ is $\inf\{y_1, y_2, y_3, \ldots\}$.
- The *upper limit* or *limsup* of $(y_1, y_2, y_3, \ldots)$ is

$$\limsup y_n = \lim_{n\to\infty} (\sup\{y_n, y_{n+1}, \ldots\}).$$

- The *lower limit* or *liminf* of $(y_1, y_2, y_3, \ldots)$ is

$$\liminf y_n = \lim_{n\to\infty}(\inf\{y_n, y_{n+1}, \ldots\}).$$

Example. If $y_n = (-1)^n(1-\frac{1}{n})$ then $\limsup y_n = 1$ and $\liminf y_n = -1$.



8.2 Series

Let X be a topological group with operation addition and let $(a_1, a_2, a_3, \ldots)$ be a sequence in X.

- The *series* $\displaystyle\sum_{n=1}^{\infty} a_n$ is the sequence $(s_1, s_2, s_3, \ldots)$,

  where $s_k = a_1 + a + 2 + \cdots + a_k$. Write $\displaystyle\sum_{n=1}^{\infty} a_n = \ell$ if $\displaystyle\lim_{n\to\infty} s_n = \ell$.

- The series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges in* X if the sequence $(s_1, s_2, s_3, \ldots)$ converges in X.

- The series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges in* X if the sequence $(s_1, s_2, s_3, \ldots)$ diverges in X.

Theorem 8.1. *(Root and ratio tests) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.*

(a) If $\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = a$ exists and $a<1$ then $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges in $\mathbb{R}$.

(b) If $\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = a$ exists and $a>1$ then $\displaystyle\sum_{n=1}^{\infty}|a_n|$ diverges in $\mathbb{R}$.

(c) If $\displaystyle\lim_{n\to\infty}|a_n|^{1/n} = a$ exists and $a<1$ then $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges in $\mathbb{R}$.

(d) If $\displaystyle\lim_{n\to\infty}|a_n|^{1/n} = a$ exists and $a>1$ then $\displaystyle\sum_{n=1}^{\infty}|a_n|$ diverges in $\mathbb{R}$.

8.3 Absolute convergence

Proposition 8.2. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{K}$.*

$$\text{If} \quad \sum_{n=1}^{\infty} |a_n| \text{ converges in } \mathbb{R}_{\geq 0} \qquad \text{then} \qquad \sum_{n=1}^{\infty} a_n \text{ converges in } \mathbb{K}.$$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{K}$.

- The series $\sum_{n=1}^{\infty} a_n$ *converges absolutely in* $\mathbb{K}$ if $\sum_{n=1}^{\infty} |a_n|$ converges in $\mathbb{R}_{\geq 0}$.
- The series $\sum_{n=1}^{\infty} a_n$ *converges conditionally in* $\mathbb{K}$ if

$$\sum_{n=1}^{\infty} |a_n| \text{ diverges in } \mathbb{R}_{\geq 0} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \text{ converges in } \mathbb{K}.$$

Proposition 8.3.

(a) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{C}$ which converges absolutely in $\mathbb{C}$.

$$\text{Let } a = \sum_{n=1}^{\infty} a_n. \qquad \text{Then} \quad \text{every rearrangement of } \sum_{n=1}^{\infty} a_n \text{ converges to } a.$$

(b) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ which converges conditionally in $\mathbb{R}$.

$$\text{If } \ell \in \mathbb{R} \quad \text{then} \quad \text{there exists a rearrangement of } \sum_{n=1}^{\infty} a_n \text{ which converges to } \ell.$$

8.4 Radius of convergence

Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{C}$ and let

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (\text{an element of } \mathbb{C}[[x]]).$$

The *radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$ is

$$\mathrm{ROC}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sup\left\{ |r| \;\middle|\; r \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} a_n r^n \text{ converges} \right\}.$$

The following proposition is what ensures that the knowledge of $\mathrm{ROC}\left(\sum_{n=0}^{\infty} a_n x^n\right)$ is useful.

Proposition 8.4. *Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Let $r, s \in \mathbb{C}$ and*

$$\text{assume } \sum_{n=0}^{\infty} a_n s^n \text{ converges.} \qquad \text{If } |r| < |s| \quad \text{then} \quad \sum_{n=0}^{\infty} a_n |r|^n \text{ converges.}$$

Proposition 8.5. *(Leibniz's theorem) If $(a_1, a_2, a_3, \ldots)$ is a decreasing sequence in $\mathbb{R}_{\ge 0}$*

$$\text{such that} \quad \lim_{n\to\infty} a_n = 0 \qquad \text{then} \qquad \sum_{n=1}^{\infty} (-1)^n a_n \text{ converges.}$$

The favorite example here is $(a_1, a_2, \ldots) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, which has

$$\sum_{i=1}^{\infty} (-1)^{n-1}\tfrac{1}{n} = \log 2 \qquad \text{and} \qquad \sum_{i=1}^{\infty} \left|(-1)^{n-1}\tfrac{1}{n}\right| = \sum_{i=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

8.5 Harmonic series and the Riemann zeta function

Let $s \in \mathbb{C}$. The *Riemann zeta function at s* is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The *harmonic series* is $\zeta(1)$. A *p-series* is $\zeta(p)$ for $p \in \mathbb{R}_{>0}$.

Theorem 8.6. *Assume $p \in \mathbb{R}_{>0}$. Then*

$$\zeta(p) \text{ converges if and only if } p \in \mathbb{R}_{>1}.$$

Proof. Case 1: $p = 1$. In this case $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

Case 2: $p \in \mathbb{R}_{<1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Case 3: $p \in \mathbb{R}_{>1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}} + \cdots \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots \\
&= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}.
\end{aligned}$$

□

1. (contractive sequences) Let Y be a metric space and let $(y_1, y_2, y_3, \ldots)$ be a sequence in Y. The sequence $(y_1, y_2, y_3, \ldots)$ is *contractive* if $(y_1, y_2, \ldots)$ satisfies: There exists $\alpha \in (0, 1)$ such that

$$\text{if } i \in \mathbb{Z}_{>0} \qquad \text{then} \quad d(y_i, y_{i+1}) \le \alpha d(y_{i-1}, y_i).$$

Show that ?????????

8.6 Some proofs

Theorem 8.7. *Let $(a_1, a_2, a_3, \dots)$ be a sequence in $\mathbb{R}$.*

(a) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ *exists and* $a < 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

(b) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ *exists and* $a > 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *diverges.*

(c) If $\lim_{n\to\infty} |a_n|^{1/n} = a$ *exists and* $a < 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

(d) If $\lim_{n\to\infty} |a_n|^{1/n} = a$ *exists and* $a > 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *diverges.*

Proof.

(a) Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a < 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon < 1$.
Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\frac{|a_{n+1}|}{|a_n|} < a + \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right) + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right)\left(\frac{|a_{N+3}|}{|a_{N+2}|}\right) + \cdots \\
&< |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}|(a+\varepsilon) + a_{N+1}(a+\varepsilon)^2 + \cdots \\
&= |a_0| + \cdots + |a_N + |a_{N+1}|(1 + (a+\varepsilon) + (a+\varepsilon)^2 + \cdots) \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| \left(\frac{1}{1-(a+\varepsilon)}\right).
\end{aligned}$$

Then, since $a + \varepsilon < 1$, $\sum_{n=0}^{\infty} |a_n|$ converges.

(b) Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a > 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon > 1$.
Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\frac{|a_{n+1}|}{|a_n|} < a - \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right) + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right)\left(\frac{|a_{N+3}|}{|a_{N+2}|}\right) + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}|(a-\varepsilon) + a_{N+1}(a-\varepsilon)^2 + \cdots \\
&> |a_0| + \cdots + |a_N| + |a_{N+1}|(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots) \\
&> |a_0| + \cdots + |a_N| + |a_{N+1}|(1 + 1 + 1 + \cdots).
\end{aligned}$$

So $\sum_{n=0}^{\infty} |a_n|$ diverges.

(c) Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a < 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon < 1$.
Since $\lim_{n\to\infty} |a_n|^{1/n} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n|^{1/n} < a + \varepsilon$.
Then

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\ &= |a_0| + \cdots + a_N| + \left(|a_{N+1}|^{1/(N+1)}\right)^{N+1} + \left(|a_{N+2}|^{1/(N+2)}\right)^{N+2} + \cdots \\ &< |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1} + (a+\varepsilon)^{N+2} + \cdots \\ &= |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1}(1 + (a+\varepsilon) + (a+\varepsilon)^2 + \cdots) \\ &= |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1}\left(\frac{1}{1-(a+\varepsilon)}\right).\end{aligned}$$

Then, since $a + \varepsilon < 1$, $\sum_{n=0}^{\infty} |a_n|$ converges.

(d) Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a > 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon > 1$.
Since $\lim_{n\to\infty} |a_n|^{1/n} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n|^{1/n} < a - \varepsilon$.
Then

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\ &= |a_0| + \cdots + |a_N| + \left(|a_{N+1}|^{1/(N+1)}\right)^{N+1} + \left(|a_{N+2}|^{1/(N+2)}\right)^{N+2} + \cdots \\ &> |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1} + (a-\varepsilon)^{N+2} + \cdots \\ &= |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1}(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots) \\ &> |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1}(1 + 1 + 1 + \cdots).\end{aligned}$$

So $\sum_{n=0}^{\infty} |a_n|$ diverges.

□

Proposition 8.8. *Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$.*

$$\text{If} \quad \sum_{n=1}^{\infty} |a_n| \text{ converges} \qquad \text{then} \qquad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof.

Assume that $\sum_{n=0}^{\infty} |a_n|$ converges.

To show: $\sum_{n=0}^{\infty} a_n$ converges.
Let $A_n = |a_0| + |a_1| + \cdots + |a_n|$ and $s_n = a_0 + a_1 + \cdots + a_n$.
Since $\sum_{n=0}^{\infty} |a_n| = (A_0, A_1, \ldots)$ converges, the sequence $(A_0, A_1, \ldots)$ is Cauchy.
Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$.
Since

$$|s_n - s_m| = |a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| = |A_n - A_m|,$$

the sequence $(s_0, s_1, \ldots)$ is Cauchy.
Since Cauchy sequences converge in $\mathbb{R}$ and $\mathbb{C}$ (in any complete metric space), the sequence $(s_0, s_1, \ldots) = \sum_{n=1}^{\infty} a_n$ converges.

□

Proposition 8.9. *Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Let $r, s \in \mathbb{C}$ and*

$$\textit{assume } \sum_{n=0}^{\infty} a_n s^n \textit{ converges.} \quad \textit{If } |r| < |s| \quad \textit{then} \quad \sum_{n=0}^{\infty} a_n |r|^n \textit{ converges.}$$

Proof.

Since $\sum_{n=0}^{\infty} a_n s^n$ converges, $\lim_{n\to\infty} |a_n s^n| = 0$.
Let $\varepsilon \in \mathbb{R}_{>0}$.
Then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n s^n| < \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n r^n| &= |a_0| + |a_1 r| + \cdots + |a_N r^N| + |a_{N+1} r^{N+1}| + \cdots \\
&= |a_0| + \cdots + |a_N r^N| + |a_{N+1} s^{N+1}| \left| \frac{r^{N+1}}{s^{N+1}} \right| + |a_{N+2} s^{N+2}| \left| \frac{r^{N+2}}{s^{N+2}} \right| + \cdots \\
&< |a_0| + \cdots + |a_N r^N| + \varepsilon \left| \frac{r^{N+1}}{s^{N+1}} \right| + \varepsilon \left| \frac{r^{N+2}}{s^{N+2}} \right| + \cdots \\
&= |a_0| + \cdots + |a_N r^N| + \varepsilon \left| \frac{r^{N+1}}{s^{N+1}} \right| \left(1 + \left|\frac{r}{s}\right| + \left|\frac{r^2}{s^2}\right| + \cdots \right) \\
&= |a_0| + |a_1 r| + \cdots + |a_N r^N| + \varepsilon \left| \frac{r^{N+1}}{s^{N+1}} \right| \left(\frac{1}{1 - \left|\frac{r}{s}\right|} \right).
\end{aligned}$$

Thus, since $|r| < |s|$, $\sum_{n=0}^{\infty} |a_n r^n|$ converges.

So, by the previous Proposition, $\sum_{n=0}^{\infty} a_n |r|^n$ converges.

□

Proposition 8.10. *(Leibniz's theorem) If $(a_1, a_2, a_3, \ldots)$ is a decreasing sequence in $\mathbb{R}_{\geq 0}$*

$$\text{such that} \quad \lim_{n\to\infty} a_n = 0 \qquad \text{then} \qquad \sum_{n=1}^{\infty} (-1)^n a_n \ \text{converges.}$$

Proof.

Assume $(a_0, a_1, \ldots)$ is a sequence in $\mathbb{R}_{\geq 0}$, $\lim_{n\to\infty} a_n = 0$ and if $n \in \mathbb{Z}_{\geq 0}$ then $a_n \geq a_{n+1}$.

To show: $\sum_{n=0}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.

Let

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}).$$

Then $s_{2m} \leq s_{2(m+1)}$.

Since $s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$, then $s_{2m} \leq a_1$.

So the sequence $(s_2, s_2, s_6, \ldots)$ is increasing and bounded above.

So $\lim_{m\to\infty} s_{2m}$ exists.

Let $\ell = \lim_{m\to\infty} s_{2m}$.

Let $s_{2m+1} = s_{2m} + a_{2m+1}$.

Then

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1} = \ell + 0 = \ell.$$

So $\lim_{m\to\infty} s_m = \ell$.

So $\sum_{n=0}^{\infty} (-1)^{n-1} a_n = \ell$.

□