9 Spaces

The point of this section is to introduce the following types of spaces and establish the following relations between these classes.

$$\left\{\begin{matrix}\text{topological}\\ \text{spaces}\end{matrix}\right\} \supseteq \left\{\begin{matrix}\text{uniform}\\ \text{spaces}\end{matrix}\right\} \supseteq \left\{\begin{matrix}\text{metric}\\ \text{spaces}\end{matrix}\right\} \supseteq \left\{\begin{matrix}\text{normed}\\ \text{vector spaces}\end{matrix}\right\} \supseteq \left\{\begin{matrix}\text{positive definite}\\ \text{inner product spaces}\end{matrix}\right\}.$$

9.1 Topological spaces

A *topological space* is a set X with a specification of the *open* subsets of X where it is required that

(a) $\emptyset$ is open in X and X is open in X,

(b) Unions of open sets in X are open in X,

(c) Finite intersections of open sets in X are open in X.

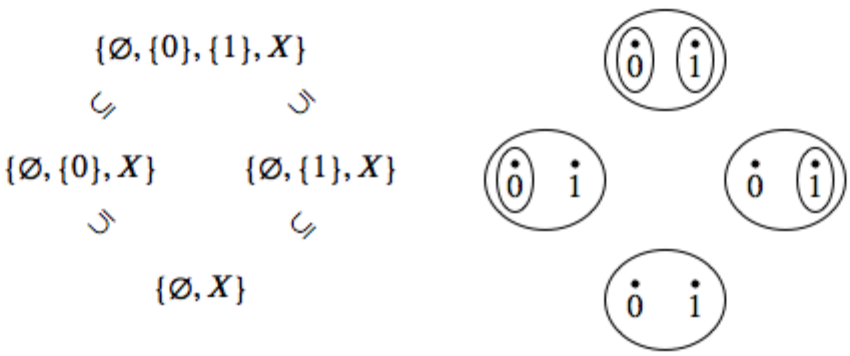
In other words, a *topology* on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U\in\mathcal{S}} U\right) \in \mathcal{T}$,

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

A *topological space* is a set X with a topology $\mathcal{T}$ on X. An *open set in* X is a set in $\mathcal{T}$.



The four possible topologies on $X = \{0, 1\}$.

9.2 Uniform spaces

Let X be a set. The set of (ordered) pairs of elements of X is

$$X \times X = \{(x_1, x_2) \mid x_1, x_2 \in X\}. \qquad \text{The } \textit{diagonal} \text{ is} \quad \Delta(X) = \{(x, x) \mid x \in X\},$$

a subset of $X \times X$. For $E \subseteq X \times X$ let

$$\sigma(E) = \{(y, x) \in X \times X \mid (x, y) \in D\}, \qquad \text{and}$$
$$E \times_X E = \{(x, y) \in X \times X \mid \text{there exists } z \in X \text{ such that } (x, z) \in E \text{ and } (z, y) \in E\}.$$

A *uniformity on* X is a collection $\mathcal{E}$ of subsets of $X \times X$ such that

(a) (diagonal condition) If $E \in \mathcal{E}$ then $\Delta(X) \subseteq E$,

(b) (upper ideal) If $E \in \mathcal{E}$ and $D \subseteq X \times X$ and $D \supseteq E$ then $D \in \mathcal{E}$,

(c) (finite intersection) If $\ell \in \mathbb{Z}_{>0}$ and $E_1, E_2, \ldots, E_\ell \in \mathcal{E}$ then $E_1 \cap E_2 \cap \cdots \cap E_\ell \in \mathcal{E}$,

(d) (symmetry condition) If $E \in \mathcal{E}$ then $\sigma(E) \in \mathcal{E}$,

(e) (triangle condition) If $E \in \mathcal{E}$ then there exists $D \in \mathcal{E}$ such that $D \times_X D \subseteq E$.

A *uniform space* is a set X with a uniformity $\mathcal{E}$ on X. An *fatdiagonal*, or *entourage*, is a set in $\mathcal{E}$.

9.3 Metric spaces

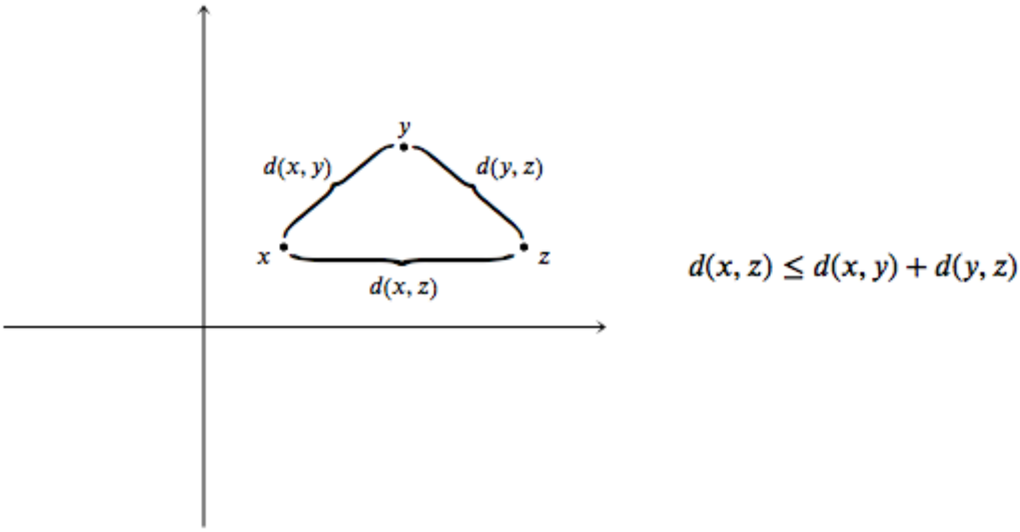
A *metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) (diagonal condition) If $x \in X$ then $d(x,x) = 0$,

(b) (diagonal condition) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$,

(c) (symmetry condition) If $x, y \in X$ then $d(x,y) = d(y,x)$,

(d) (the triangle inequality) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$.



Distances between points in $\mathbb{R}^2$.

9.4 Normed vector spaces

Let $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ with $i^2 = -1$ be the field of complex numbers with complex conjugation

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ c & \mapsto & \overline{c} \end{array} \qquad \text{given by} \qquad \overline{a+bi} = a - bi,$$

and absolute value

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{R}_{\geq 0} \\ c & \mapsto & |c| \end{array} \qquad \text{given by} \qquad |c|^2 = c\,\overline{c}.$$

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-*vector space* is a set V with functions

$$\begin{array}{ccc} V\times V & \to & V \\ (v_1,v_2) & \mapsto & v_1+v_2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{K}\times V & \to & V \\ (c,v) & \mapsto & cv \end{array}$$

(*addition* and *scalar multiplication*) such that

(a) If $v_1,v_2,v_3\in V$ then $(v_1+v_2)+v_3=v_1+(v_2+v_3)$,

(b) There exists $0\in V$ such that if $v\in V$ then $0+v=v$ and $v+0=v$,

(c) If $v\in V$ then there exists $-v\in V$ such that $v+(-v)=0$ and $(-v)+v=0$,

(d) If $v_1,v_2\in V$ then $v_1+v_2=v_2+v_1$,

(e) If $c\in\mathbb{K}$ and $v_1,v_2\in V$ then $c(v_1+v_2)=cv_1+cv_2$,

(f) If $c_1,c_2\in\mathbb{K}$ and $v\in V$ then $(c_1+c_2)v=c_1v+c_2v$,

(g) If $c_1,c_2\in\mathbb{K}$ and $v\in V$ then $c_1(c_2v)=(c_1c_2)v$,

(h) If $v\in V$ then $1v=v$.

A *normed vector space* is a $\mathbb{K}$-vector space V with a function $\|\ \|\colon V\to\mathbb{R}_{\geq 0}$ such that

(a) If $x,y\in V$ then $\|x+y\|\leq\|x\|+\|y\|$,

(b) If $c\in\mathbb{K}$ and $v\in V$ then $\|cv\|=|c|\,\|v\|$,

(c) If $v\in V$ and $\|v\|=0$ then $v=0$.

9.5 Inner product spaces

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

A *positive definite symmetric inner product space* is a $\mathbb{K}$-vector space V with a function

$$\begin{array}{ccc} V\times V & \to & \mathbb{K} \\ (v_1,v_2) & \mapsto & \langle v_1,v_2\rangle \end{array} \qquad \text{such that}$$

(a) (symmetry condition) If $v_1,v_2\in V$ then $\langle v_1,v_2\rangle=\langle v_2,v_1\rangle$,

(b) (linearity in the first coordinate) If $c_1,c_2\in\mathbb{K}$ and $v_1,v_2,v_3\in V$ then $\langle c_1v_1+c_2v_2,v_3\rangle = c_1\langle v_1,v_3\rangle+c_2\langle v_2,v_3\rangle$,

(c) (linearity in the second coordinate) If $c_1,c_2\in\mathbb{K}$ and $v_1,v_2,v_3\in V$ then $\langle v_3,c_1v_1+c_2v_2\rangle = c_1\langle v_3,v_1\rangle+c_2\langle v_3,v_2\rangle$,

(d) (no isotropic vectors condition) If $v\in V$ and $\langle v,v\rangle=0$ then $v=0$.

(e) (positive definite condition) If $v\in V$ then $\langle v,v\rangle\in\mathbb{R}_{\geq 0}$.

A *positive definite Hermitian inner product space* is a $\mathbb{K}$-vector space V with a function

$$\begin{array}{ccc} V\times V & \to & \mathbb{K} \\ (v_1,v_2) & \mapsto & \langle v_1,v_2\rangle \end{array} \qquad \text{such that}$$

(a) (symmetry condition) If $v_1, v_2 \in V$ then $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$,

(b) (linearity in the first coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle c_1v_1 + c_2v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle$,

(c) (conjugate linearity in the second coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle v_3, c_1v_1 + c_2v_2\rangle = \overline{c_1}\langle v_3, v_1\rangle + \overline{c_2}\langle v_3, v_2\rangle$,

(d) (no isotropic vectors condition) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

(e) (positive definite condition) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$.

An *inner product space* is a positive definite symmetric inner product space or a positive definite Hermitian inner product space.

9.6 Uniform spaces can be made into topological spaces

Let $(X, \mathcal{E})$ be a uniform space.

Let $E \in \mathcal{E}$ and $x \in X$. The *E-neighborhood of x* is

$$B_E(x) = \{y \in X \mid (x, y) \in E\}.$$

Let $x \in X$. The *neighborhood filter* of x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } E \in \mathcal{X} \text{ such that } N \supseteq B_E(x)\}.$$

The *uniform space topology* on X is the topology

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } E \in \mathcal{E} \text{ such that } B_E(x) \subseteq U\}.$$

9.7 Metric spaces can be made into topological spaces, and into uniform spaces

A tolerance is a number of decimal places of accuracy to achieve in a measurement. The *set of tolerances* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

Let (X, d) be a metric space.

- Let $x \in X$ and $\epsilon \in \mathbb{E}$. The *open ball of radius ϵ at x* is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

- Let $\epsilon \in \mathbb{E}$. The *diagonal of width ϵ*, or *ϵ-diagonal*, is

$$B_\epsilon = \{(y, x) \in X \times X \mid d(x, y) < \epsilon\}.$$

Let $x \in X$. The *neighborhood filter of x* is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } \epsilon \in \mathbb{E} \text{ such that } N \supseteq B_\epsilon(x)\}.$$

The *metric space topology on X* is

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The *metric space uniformity on* X is

$$\mathcal{E} = \{\text{subsets of } X \times X \text{ which contain an } \epsilon\text{-diagonal}\}.$$

More precisely, $E \subseteq X \times X$ is a *fatdiagonal in* X if and only if

$$\text{there exists } \epsilon \in \mathbb{E} \text{ such that} \quad E \supseteq B_\epsilon.$$

Proposition 9.1. *Let* (X, d) *be a metric space. Let*

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0} \text{ and } x \in X\},$$

the set of open balls in X*. Let* $\mathcal{T}$ *be the metric space topology on* X*. Then* U *is an open set in* X *if and only if*

$$\text{there exists } \mathcal{S} \subseteq \mathcal{B} \text{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B,$$

9.8 $\{$normed vector spaces$\} \subseteq \{$metric spaces$\}$

Let $(V, \|\ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d \colon V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

9.9 $\{$inner product spaces$\} \subseteq \{$normed vector spaces$\}$

Let $(V, \langle, \rangle)$ be a positive definite symmetric inner product space or a positive definite Hermitian inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v \rangle.$$

9.10 Notes and references

On the face of it, it might look like there are no proofs or Propositions in this chapter, but this is not the case at all. It is necessary and important to prove carefully that

(a) the uniform space topology is a topology,

(b) the metric space topology is a topology,

(c) the metric space uniformity is a uniformity,

(d) the norm metric is a metric, and

(e) the length norm is a norm.

Fortunately, for a practiced "proof machine" user these proofs are straightforward. For beginners at "proof machine", these are excellent homework assignment (and exam) questions.

The definition of uniform spaces in Section 9.2 follows [Bou, Top. Ch. II]. It is structured to model and highlight the analogies to topological spaces, and to provide a bridge between topological spaces and metric spaces. It is helpful to remember that the elements of a uniformity are called "entourages", in the same way that the elements of a topology are called "open sets". **The category of uniform spaces is the natural home for uniformly continuous functions, Cauchy sequences and completion.** Uniformly continuous functions are introduced in Chapter 11.2 and Cauchy sequences and completion are discussed in Chapter **??**.

To relate the definitions of a uniform space and a metric space it is helful to note that conditions (a) and (b) in the definition of a metric space are equivalent to $d^{-1}(0) = \Delta_X$. A uniform space is almost a metric space since every uniformity can be obtained as the supremum of uniformities coming from pseudometrics. By [Bou, Top. Ch. IX §2 no. 4 Theorem 1] the separable Hausdorff uniform spaces are exactly the separable metric spaces. The condition for a topological space to be a uniform space is given in [Bou, Top. Ch. IX §1 no. 5 Theorem 2] (see also the discussion at the beginning of [Bou, Top. Ch. II §4 no. 1]). By [Bou, Top. Ch. II §1 no. 2 Cor. 3 to Prop. 2], a topological space that does not satisfy axiom ($\mathrm{O}_{\mathrm{III}}$) of [Bou, Top. Ch. I §8 no. 4] is not uniformizable.

In practice, it is often more convenient to work with a good set of *generators* of a topology rather than with *all* the sets in a topology. Let $(X, \mathcal{T})$ be a topological space. A *union generating set for* $\mathcal{T}$, or a *basis of* $\mathcal{T}$, is a collection $\mathcal{B}$ of subsets of X such that

$$\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}.$$

The collection of open balls of radius ϵ centered at x is a union generating set for the metric space topology of a metric space (X, d).

In exact analogy to the case of topological spaces, it is often more convenient to work with a good set of *generators* of a uniformity rather than with *all* the sets in a uniformity. Let $(X, \mathcal{X})$ be a uniform space. An *inclusion generating set for* $\mathcal{X}$ is a collection $\mathcal{D}$ of subsets of $X\times$ such that

$$\mathcal{X} = \{\text{subsets of } X \times X \text{ that contain a set in } \mathcal{D}\}.$$

The collection of ϵ-diagonals is an inclusion generating set for the metric space uniformity of a metric space (X, d).

In history of the development of theory of topological spaces, the question of when a topological space X is a metric space was an important and motivating problem. Some initial insight into this question is provided by the exercises in Section **??**. A complete answer is given in Exercises (15-17) in Section 18.2. These results sometimes go under the name "The Urysohn metrization theorem" (see https://terrytao.wordpress.com/2009/03/18/245b-notes-13-compactification-and-metrisation-optional/#more-1901).

In the definition of inner products the diagonal conditions are anisotropy conditions. The norm condition is necessary for $\|v\| = \sqrt{\langle v, v\rangle}$ to be a norm.