41 Tutorial 8: Uniform spaces and uniform continuity

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

41.1 Uniformly continuous functions

Proposition 41.1.

(a) Let $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$ and $(Z,\mathcal{T}_Z)$ be topological spaces and let $f\colon X\to Y$ and $g\colon Y\to Z$ be continuous functions. Then

$$g\circ f\colon X\to Z \quad \textit{is a continuous function.}$$

(b) Let $(X,\mathcal{E}_X)$, $(Y,\mathcal{E}_Y)$ and $(Z,\mathcal{E}_Z)$ be uniform spaces and let $f\colon X\to Y$ and $g\colon Y\to Z$ be uniformly continuous functions. Then

$$g\circ f\colon X\to Z \quad \textit{is a uniformly continuous function.}$$

Proof. (a) To show: If $V\in\mathcal{T}_Z$ then $(g\circ f)^{-1}(V)\in\mathcal{T}_X$.
Assume $V\in\mathcal{T}_Z$.
Since g is continuous then $g^{-1}(V)\in\mathcal{T}_Y$.
Since f is continuous then $f^{-1}(g^{-1}(V))\in\mathcal{T}_X$.
So

$$\begin{aligned}
f^{-1}(g^{-1}(V)) &= \{x\in X \mid f(x)\in g^{-1}(V)\}\\
&= \{x\in X \mid g(f(x))\in V\}\\
&= \{x\in X \mid (g\circ f)(x))\in V\}\\
&= (g\circ f)^{-1}(V) \text{ is an element of } \mathcal{T}_X.
\end{aligned}$$

So $g\circ f$ is continuous.

(b) To show: If $V\in\mathcal{E}_Z$ then $((g\circ f)\times(g\circ f))^{-1}(V)\in\mathcal{E}_X$.
Assume $V\in\mathcal{E}_Z$.
Since g is uniformly continuous then $(g\times g)^{-1}(V)\in\mathcal{E}_Y$.
Since f is uniformly continuous then $(f\times f)^{-1}((g\times g)^{-1}(V))\in\mathcal{E}_X$.

So

$$\begin{aligned}(f\times f)^{-1}((g\times g)^{-1}(V)) &= \{(x_1,x_2)\in X\times X \mid f(x_1),f(x_2))\in (g\times g)^{-1}(V)\}\\ &= \{(x_1,x_2)\in X\times X \mid (g(f(x_1)),g(f(x_2)))\in V\}\\ &= \{(x_1,x_2)\in X\times X \mid ((g\circ f)(x_1),(g\circ f)(x_2))\in V\}\\ &= \{(x_1,x_2)\in X\times X \mid ((g\circ f)\times(g\circ f))(x_1,x_2))\in V\}\\ &= ((g\circ f)\times(g\circ f))^{-1}(V) \text{ is an element of } \mathcal{E}_X.\end{aligned}$$

So $g\circ f$ is uniformly continuous.

□

41.2 Uniformly continuous functions are continuous

Proposition 41.2. *Let $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ be uniform spaces. Let $\mathcal{T}_X$ be the uniform space topology on $(X,\mathcal{E}_X)$ and let $\mathcal{T}_Y$ be the uniform space topology on $(Y,\mathcal{E}_Y)$.*

If $f\colon X\to Y$ is uniformly continuous then $f\colon X\to Y$ is continuous.

Proof. Assume $f\colon X\to Y$ is uniformly continuous.

To show: $f\colon X\to Y$ is continuous.
To show: If $a\in A$ then $f\colon X\to Y$ is continuous at a.
Assume $a\in X$.
To show: f is continuous at a.

To show: If $V\in\mathcal{N}(f(a))$ then $f^{-1}(V)\in\mathcal{N}(a)$.
Assume $V\in\mathcal{N}(f(a))$.
To show: $f^{-1}(V)\in\mathcal{N}(a)$.

To show: There exists $D\in\mathcal{E}_X$ such that $f^{-1}(V)\supseteq B_D(a)$.
Since $V\in\mathcal{N}(f(a))$ there exists $C\in\mathcal{E}_Y$ such that $V\supseteq B_C(f(a))$.
Let $D=(f\times f)^{-1}(C)$.
To show: $f^{-1}(V)\supseteq B_D(a)$.
To show: If $y\in B_D(a)$ then $y\in f^{-1}(V)$.

Assume $y\in B_D(a)$.
Then $(a,y)\in D$.
So $(a,y)\in(f\times f)^{-1}(C)$.
So $(f(a),f(y))\in C$.
So $f(y)\in B_C(f(a))$.
So $f(y)\in V$.
So $y\in f^{-1}(V)$.

So $f^{-1}(V)\supseteq B_D(a)$.

So $f^{-1}(V)\in\mathcal{N}(a)$.

So f is continuous at a.

So f is continuous. □

41.3 epsilon-delta characterization of continuity and uniform continuity

Proposition 41.3. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function.*

(a) The function $f : X \to Y$ is continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ and } a \in X \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } x \in X \textit{ and } d_X(a, x) < \delta \textit{ then } d_Y(f(a), f(x)) < \epsilon.\end{aligned}$$

(b) The function $f : X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } a, x \in X \textit{ and } d_X(a, x) < \delta \textit{ then } d_Y(f(a), f(x)) < \epsilon.\end{aligned}$$

Proof. (a) $\Rightarrow$: Assume $f : X \to Y$ is continuous.

To show: If $\epsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$ and $a \in X$.
To show: There exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Since f is continuous and $B_\epsilon(f(a))$ is open in Y then $f^{-1}(B_\epsilon(f(a)))$ is open in X.
Using that $f^{-1}(B_\epsilon(f(a)))$ is open in X and $a \in f^{-1}(B_\epsilon(f(a)))$,

$$\text{let } \delta \in \mathbb{E} \quad \text{such that} \quad B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a))).$$

To show: If $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $x \in X$ and $d_X(a, x) < \delta$.
To show: $d_Y(f(a), f(x)) < \epsilon$.
Since $x \in B_\delta(a)$ and $B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a)))$ then

$$f(x) \in B_\epsilon(f(a)).$$

So $d_Y(f(a), f(x)) < \epsilon$.

(a) $\Leftarrow$: Assume: If $\epsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

To show: $f : X \to Y$ is continuous.

Let $\mathcal{T}_X$ be the metric space topology for (X, d_X).
Let $\mathcal{T}_Y$ be the metric space topology for (Y, d_Y).

To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
Then $f(a) \in V$ and, since V is open in Y,

$$\text{there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(f(a)) \subseteq V.$$

To show: a is an interior point of $f^{-1}(V)$.
To show: There exists $\gamma \in \mathbb{E}$ such that $B_\gamma(a) \subseteq f^{-1}(V)$.
We know there exists $\delta \in \mathbb{E}$ such that 'if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$'.

Let $\gamma = \delta$.
To show: $B_\gamma(a) \subseteq f^{-1}(V)$.
Since δ satisfies 'if $x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$',

$$\text{then} \quad f(B_\gamma(a)) = f(B_\delta(a)) \subseteq B_\epsilon(f(a)).$$

Since $B_\epsilon(f(a)) \subseteq V$ then $B_\gamma(a) \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So $f : X \to Y$ is continuous.

(b) $\Rightarrow$: Assume $f : X \to Y$ is uniformly continuous.

To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

Let $\mathcal{E}_X$ be the metric space uniformity for (X, d_X).
Let $\mathcal{E}_Y$ be the metric space uniformity for (Y, d_Y).

Assume $\epsilon \in \mathbb{E}$.
To show: There exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Since f is uniformly continuous and $B_\epsilon \in \mathcal{E}_Y$ then $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$.
Since $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$, then

$$\text{there exists } \gamma \in \mathbb{E} \quad \text{such that} \quad B_\gamma \subseteq (f \times f)^{-1}(B_\epsilon).$$

Let $\delta = \gamma$.
To show: If $a, x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $a, x \in X$ and $d_X(a,x) < \delta$.
To show: $d_Y(f(a), f(x)) < \epsilon$.
Since $d_X(a,x) < \delta = \gamma$

$$\text{then} \quad (a,x) \in B_\gamma \subseteq (f \times f)^{-1}(B_\epsilon).$$

So $(f(a), f(x)) \in B_\epsilon$.
So $d_Y(f(a), f(x)) < \epsilon$.

(b) $\Leftarrow$: Assume: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

To show: $f : X \to Y$ is uniformly continuous.

Let $\mathcal{E}_X$ be the metric space uniformity for (X, d_X).
Let $\mathcal{E}_Y$ be the metric space uniformity for (Y, d_Y).

To show: If $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Assume $E \in \mathcal{E}_Y$.
To show: $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
To show: There exists $\gamma \in \mathbb{E}$ such that $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since $E \in \mathcal{E}_Y$

$$\text{there exists } \epsilon \in \mathbb{E} \quad \text{such that} \quad B_\epsilon \subseteq E.$$

So there exists $\delta \in \mathbb{E}$ such that

$$\text{if } a, x \in X \text{ and } d_X(a,x) < \delta \text{ then } d_Y(f(a), f(x)) < \epsilon. \qquad (*)$$

Let $\gamma = \delta$.
To show: $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since δ satisfies (*), then if $(a,x) \in B_\delta = B_\gamma$ then $(f(a), f(x)) \in B_\epsilon$.
So $(f \times f)(B_\gamma) \subseteq B_\epsilon \subseteq E$.
So $B_\gamma \subseteq (f \times f)^{-1}(E)$.
So $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
So $f\colon X \to Y$ is uniformly continuous.

$\square$

Proposition 41.4. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces and let $f\colon X \to Y$ be a function.*

(a) The function $f\colon X \to Y$ is continuous if and only if f satisfies

$$\begin{gathered}\textit{if } E \in \mathcal{E}_Y \textit{ and } a \in X \textit{ then there exists } D \in \mathcal{E}_X \textit{ such that}\\ \textit{if } x \in X \textit{ and } (a,x) \in D \textit{ then } (f(a), f(x)) \in E.\end{gathered}$$

(b) The function $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{gathered}\textit{if } E \in \mathcal{E}_Y \textit{ then there exists } D \in \mathcal{E}_X \textit{ such that}\\ \textit{if } a, x \in X \textit{ and } (a,x) \in D \textit{ then } (f(a), f(x)) \in E.\end{gathered}$$

Proof. (a) $\Rightarrow$: Assume $f\colon X \to Y$ is continuous.

To show: If $E \in \mathcal{E}_Y$ and $a \in X$ then there exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a,x) \in D$ then $(f(a), f(x)) \in E$.
Assume $E \in \mathcal{E}_Y$ and $a \in X$.
To show: There exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a,x) \in D$ then $(f(a), f(x)) \in E$.
Since f is continuous and $B_E(f(a))$ is open in Y then $f^{-1}(B_E(f(a)))$ is open in X.
Using that $f^{-1}(B_E(f(a)))$ is open in X and $a \in f^{-1}(B_E(f(a)))$,

$$\text{let } D \in \mathcal{E}_X \quad \text{such that} \quad B_D(a) \subseteq f^{-1}(B_E(f(a))).$$

To show: If $x \in X$ and $(a,x) \in D$ then $(f(a), f(x)) \in E$.
Assume $x \in X$ and $(a,x) \in D$.
To show: $(f(a), f(x)) \in E$.
Since $x \in B_D(a)$ and $B_D(a) \subseteq f^{-1}(B_E(f(a)))$ then $f(x) \in B_E(f(a))$.
So $(f(a), f(x)) \in E$.

(a) $\Leftarrow$: Assume: If $E \in \mathcal{E}_Y$ and $a \in X$ then there exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a,x) \in D$ then $(f(a), f(x)) \in E$.

To show: $f\colon X \to Y$ is continuous.

Let $\mathcal{T}_X$ be the uniform space topology for $(X, \mathcal{E}_X)$.
Let $\mathcal{T}_Y$ be the uniform space topology for $(Y, \mathcal{E}_Y)$.

To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.

To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
Then $f(a) \in V$ and, since V is open in Y,

$$\text{there exists } E \in \mathcal{E}_Y \text{ such that } B_E(f(a)) \subseteq V.$$

To show: a is an interior point of $f^{-1}(V)$.
To show: There exists $G \in \mathcal{E}_X$ such that $B_G(a) \subseteq f^{-1}(V)$.
We know there exists $D \in \mathcal{E}_X$ such that 'if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$'.
Let $G = D$.
To show: $B_G(a) \subseteq f^{-1}(V)$.
Since D satisfies 'if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$',

$$\text{then} \quad f(B_G(a)) = f(B_D(a)) \subseteq B_E(f(a)).$$

Since $B_E(f(a)) \subseteq V$ then $B_G(a) \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So $f : X \to Y$ is continuous.
So $f : X \to Y$ is continuous.

(b) $\Rightarrow$: Assume $f : X \to Y$ is uniformly continuous.

To show: If $E \in \mathcal{E}_Y$ then there exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $E \in \mathcal{E}_Y$.
To show: There exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Since f is uniformly continuous and $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Since $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$, then

$$\text{there exists } G \in \mathcal{E}_X \quad \text{such that} \quad G \subseteq (f \times f)^{-1}(E).$$

Let $D = G$.
To show: If $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $a, x \in X$ and $(a, x) \in D$.
To show: $(f(a), f(x)) \in E$.
Since $(a, x) \in D = G$

$$\text{then} \quad (a, x) \in G \subseteq (f \times f)^{-1}(E).$$

So $(f(a), f(x)) \in E$.
So $d_Y(f(a), f(x)) \in E$.

(b) $\Leftarrow$: Assume: If $E \in \mathcal{E}_Y$ then there exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.

To show: $f : X \to Y$ is uniformly continuous.
To show: If $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Assume $E \in \mathcal{E}_Y$.
To show: $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
To show: There exists $G \in \mathcal{E}_X$ such that $G \subseteq (f \times f)^{-1}(E)$.

Since $E \in \mathcal{E}_Y$ there exists $D \in \mathcal{E}_X$ such that

$$\text{if } a, x \in X \text{ and } (a, x) \in D \text{ then } (f(a), f(x)) \in E. \qquad (*)$$

Let $G = D$.
To show: $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since D satisfies (*), then if $(a, x) \in D = G$ then $(f(a), f(x)) \in E$.
So $(f \times f)(G \subseteq E$.
So $G \subseteq (f \times f)^{-1}(E)$.
So $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
So $f : X \to Y$ is uniformly continuous.

$\square$

41.4 The neighborhood filter of a uniform space

Proposition 41.5. *Let $(X, \mathcal{E})$ be a uniform space. Let $x \in X$ and let $\mathcal{N}(x)$ be the neighborhood filter of x for the uniform space topology. Then*

$$\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}.$$

Proof. To show: $\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}$.
By definition $\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } E \in \mathcal{E} \text{ such that } N \supseteq B_E(x)\}$.
To show: (a) $\mathcal{N}(x) \supseteq \{B_E(x) \mid E \in \mathcal{E}\}$.
(b) $\mathcal{N}(x) \subseteq \{B_E(x) \mid E \in \mathcal{E}\}$.

(a) This is direct from the definition of $\mathcal{N}(x)$.
(b) To show: If $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{E}$ such that $N = B_W(x)$.
Assume $N \in \mathcal{N}(x)$.
Then there exists $E \in \mathcal{E}$ with $N \supseteq B_E(x)$.
To show: There exists $W \in \mathcal{E}$ such that $N = B_W(x)$.
Let $W = \{(y, x) \mid y \in N\}$.
If $(y, x) \in E$ then $y \in B_E(x) \subseteq N$ and so $(y, x) \in W$.
Thus $W \supseteq E$.
Since $E \in \mathcal{E}$ and $W \subseteq X \times X$ and $W \supseteq E$ then $W \in \mathcal{E}$.
To show: $N = B_W(x)$.
To show: (a) $N \subseteq B_W(x)$.
(b) $B_W(x) \subseteq N$.

(a) Assume $n \in N$.
Then $(n, x) \in W$ and $n \in B_W(x)$.
So $N \subseteq B_W(x)$.
(b) Assume $y \in B_W(x)$.
Then $(y, x) \in W$.
Thus, by the definition of W, $y \in N$.
So $B_W(x) \subseteq N$.

So $N = B_W(x)$.

So $\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}$. $\square$