42 Tutorial 9: Properties of the real numbers

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

42.1 Relations between $\mathbb{Q}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$

Proposition 42.1. *(The function $\iota\colon \mathbb{Q}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not surjective)*

(a) There exists $z \in \mathbb{R}_{\geq 0}$ such that $z^2 = 2$.

(b) If $z \in \mathbb{R}_{\geq 0}$ and $z^2 = 2$ then $z \notin \mathbb{Q}_{\geq 0}$.

Proof. (Sketch)
(a) Noting that $1^2 = 1 < 2$ and $2^2 = 4 > 2$, let $z_1 = 1$.
Noting that $14^2 = 196 < 200$ and $15^2 = 225 > 200$, let $z_2 = 1.4$.
Noting that $141^2 = 19881 < 20000$ and $142^2 = 20164 > 20000$, let $z_3 = 1.41$.
In general, for $k \in \mathbb{Z}_{\geq 0}$ let $a_k \in \mathbb{Z}_{>0}$ be maximal such that $a_k^2 < 2 \cdot 10^{2k}$ and let $z_{k+1} = 10^{-k} a_k$.
Then $z = (z_1, z_2, \ldots) \in \mathbb{R}_{\geq 0}$ and $z^2 = 2$.
(b) If $z = p/q \in \mathbb{Q}_{\geq 0}$ with p/q in reduced form then $2q^2 = p^2$ which implies 2 divides p which implies 2 divides q, which is a contradiction to p/q being reduced. THIS HEAVILY USES THE FACT THAT $\mathbb{Z}$ IS A UNIQUE FACTORIZATION DOMAIN. DO YOU KNOW HOW TO PROVE THAT $\mathbb{Z}$ IS A UNIQUE FACTORIZATION DOMAIN?? □

Proposition 42.2. *($\mathbb{Q}_{\geq 0}$ and the order on $\mathbb{R}_{\geq 0}$)*

(a) If $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$ then there exists $c \in \mathbb{Q}_{\geq 0}$ such that $a < c < b$.

(b) If $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$ then there exists $c \in (\mathbb{R}_{\geq 0} - \mathbb{Q}_{\geq 0})$ such that $a < c < b$.

Proof.

(a) If $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$.
To show: There exists $c \in \mathbb{Q}_{\geq 0}$ such that $a < c < b$.
Let $x \in \mathbb{R}_{\geq 0}$ such that $b = a + x$.
Let $k \in \mathbb{Z}_{>0}$ such that $10^{-k} < x$
(i.e. if $x = z.d_1d_2d_3\ldots$ then let $n \in \mathbb{Z}_{>0}$ such that $d_n \neq 0$ and let $k = n + 1$).
Let $c = a + 10^{-k}$.
Since $a < a + 10^{-k} < a + x = b$ then $a < c < b$.

(b) Since $\sqrt{2} \in \mathbb{R}_{\geq 0} - \mathbb{Q}_{\geq 0}$ then $c \in \mathbb{R}_{\geq 0} - \mathbb{Q}_{\geq 0}$.
Let $x \in \mathbb{R}_{\geq 0}$ such that $b = a + x$.
Let $k \in \mathbb{Z}_{>0}$ such that $10^{-k} < x$
Let $c = a + 10^{-k}\frac{\sqrt{2}}{2}$.
Since $a < a + 10^{-k}\frac{\sqrt{2}}{2} < a + 10^{-k} < a + x = b$ then $a < c < b$.

□

42.2 Archimedes' property and the least upper bound property

Theorem 42.3. *(Archimedes' property)*
If $x, y \in \mathbb{R}_{\geq 0}$ then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $y < nx$.

Proof. Assume $x, y \in \mathbb{R}_{>0}$.
To show: There exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.
Using Proposition 17.7(a), there exist $\frac{p}{q} \in \mathbb{Q}_{>0}$ and $\frac{r}{s} \in \mathbb{Q}_{>0}$ such that

$$0 < \frac{p}{q} < x \qquad \text{and} \qquad y < \frac{r}{s}.$$

Let $n \in \mathbb{Z}_{>0}$ be such that $nps > qr$.
Then

$$y < \frac{rq}{sq} < \frac{nsp}{sq} = nx.$$

□

Theorem 42.4. *(The least upper bound property)*
(a) If $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists in $\mathbb{R}_{\geq 0}$.

Proof. (Sketch) If $a = zd_1d_2d_3\ldots$ is the decimal expansion of a and $k \in \mathbb{Z}_{>0}$ then let

$$a_k = z.d_1d_2\cdots d_k \in \mathbb{Q}_{\geq 0} \qquad \text{(this is the kth element of the sequence corresponding to a).}$$

For $k \in \mathbb{Z}_{>0}$, define

$$A_k = \{a_k \mid a \in A\} \qquad \text{so that} \qquad A_k \subseteq \mathbb{Q}_{\geq 0} \quad \text{and} \qquad \mathrm{Card}(A_k) \leq 10^k.$$

Fro $k \in \mathbb{Z}_{>0}$ let Let

$$z_k = \max(A_k), \qquad \text{and let} \qquad z = (z_1, z_2, \ldots).$$

Check that $z = (z_1, z_2, \ldots)$ is a Cauchy sequence in $\mathbb{Q}_{\geq 0}$ and then check the defining conditions for $\sup(A)$ to complete the proof that the element of $\mathbb{R}_{\geq 0}$ given by the Cauchy sequence $z = (z_1, z_2, \ldots)$ is $\sup(A)$. □

42.3 Convergence and continuity in $\mathbb{R}_{\geq 0}$

Proposition 42.5.
(a) If $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, \ldots)$ converges to $\sup\{a_1, a_2, \ldots\}$.
(b) $\overline{\mathbb{Q}_{\geq 0}} = \mathbb{R}_{\geq 0}$.

Proof.

(a) Let $(a_1, a_2, \dots)$ be a sequence in $\mathbb{R}$ such that $a_1 \le a_2 \le \cdots$ and there exists $b \in \mathbb{R}$ such that if $i \in \mathbb{Z}_{>0}$ then $a_i < b$.
By the least upper bound property (Proposition 17.9), since $A = \{a_1, a_2, \dots\}$ is bounded then $\sup\{a_1, a_2, \dots\}$ exists.
Let $c = \sup\{a_1, a_2, \dots\}$.
To show: $\lim_{n\to\infty} a_n = c$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge \ell}$ then $|c - a_n| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge \ell}$ then $|c - a_n| < \epsilon$.
Using that $c - \epsilon$ is not an upper bound, let $\ell \in \mathbb{Z}_{>0}$ be such that $a_\ell > c - \epsilon$.
If $n \in \mathbb{Z}_{\ge \ell}$ then $a_n \ge a_\ell$ and so $c - a_n \le c - a_\ell < \epsilon$.
So $\lim_{n\to\infty} a_n = c$.
So $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, \dots\}$.

(b) Let $x = z.d_1 d_2 \dots \in \mathbb{R}_{\ge 0}$.
Let $x_k = z.d_1 d_2 \dots d_k$ be the first k decimal places of x.
Then $(x_1, x_2, \dots)$ is a sequence in $\mathbb{R}_{\ge 0}$ such that $\lim_{k\to\infty} x_k = x$.
So $\mathbb{R}_{\ge 0} \subseteq \overline{\mathbb{Q}_{\ge 0}}$.
Since $\overline{\mathbb{Q}_{\ge 0}}$ means closure of $\mathbb{Q}_{\ge 0}$ in $\mathbb{R}_{\ge 0}$ then $\overline{\mathbb{Q}_{\ge 0}} \subseteq \mathbb{R}_{\ge 0}$.
So $\overline{\mathbb{Q}_{\ge 0}} = \mathbb{R}_{\ge 0}$.

□

Theorem 42.6. *Let $n \in \mathbb{Z}_{>0}$. The function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous, bijective, and satisfies*

$$\text{if } x, y \in \mathbb{R}_{\ge 0} \text{ and } x < y \quad \text{then} \quad x^n < y^n.$$

Furthermore, the inverse function $x^{\frac{1}{n}} \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous.

Proof. (Sketch)
To show: (a) The function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is monotone.
(b) The function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is injective.
(c) The function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is surjective.
(d) The function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous.
(e) The inverse function $x^{1/n} \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ exists and is continuous.

(a) Assume $x, y \in \mathbb{R}_{\ge 0}$ and $x < y$.
Then there exists $z \in \mathbb{R}_{\ge 0}$ such that $x + z = y$.
Using the binomial theorem,

$$x^n < x^n + z^n < x^n + \left(\sum_{j=1}^{n-1} \binom{n}{j} x^{n-j} y^j\right) + y^n = (x+z)^n = y^n.$$

So the function $x^n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is monotone.

(b) Assume $x, y \in \mathbb{R}_{\ge 0}$ and $x \ne y$.
By (a), if $x < y$ then $x^n < y^n$ and $x^n \ne y^n$ and if $x > y$ then $x^n > y^n$ and $x^n \ne y^n$.

So the function $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ then the function $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is injective.

(c) To show: The function $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is surjective.
To show: If $z \in \mathbb{R}_{\ge 0}$ then there exists $x \in \mathbb{R}_{\ge 0}$ such that $x^n = z$.
Assume $z \in \mathbb{R}_{\ge 0}$.
By the least upper bound property (Proposition 17.9), $z = \sup\{y \in \mathbb{R}_{\ge 0} \mid y^n < x\}$ exists in $\mathbb{R}_{\ge 0}$.
Then $z^n = x$.
So the function $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is surjective.

(d) To show: If $a \in \mathbb{R}_{\ge 0}$ then $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous at a.
Assume $a \in \mathbb{R}_{\ge 0}$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $y \in \mathbb{R}_{\ge 0}$ and $d(y, a) < \delta$ then $d(y^n, a^n) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
To show: There exists $\delta \in \mathbb{E}$ such that if $d(y, a) < \delta$ then $d(y^n, a^n) < \epsilon$.
Let $\delta = \frac{1}{2^n a^{n-1}}\epsilon$.
Letting $d = d(a, y)$ then

$$d(y^n, a^n) = |y^n - a^n| = |(a+d)^n - a^n| = da^{n-1} \cdot \Big(\sum_{j=1}^{n} \frac{d^{j-1}}{a^{j-1}} \binom{n}{j}\Big)$$
$$< da^{n-1}\Big(\sum_{j=1}^{n} \binom{n}{j}\Big) = da^{n-1}(2^n - 1) < \delta 2^n a^{n-1} = \epsilon.$$

(What is at the core of this is that the distance $d(y^n, a^n)$ is related to the distance $d(y, a)$ by

$$d(y, a)a^{n-1}n < d(y^n, a^n) < d(y, a)a^{n-1}(2^n - 1).$$

So $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous at a.
So $x^n\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous.

(e) To show: If $b \in \mathbb{R}_{\ge 0}$ then $x^{\frac{1}{n}}\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous at b.
Assume $b \in \mathbb{R}_{\ge 0}$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $z \in \mathbb{R}_{\ge 0}$ and $d(z, b) < \delta$ then $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
Let $\delta = na^{n-1}\epsilon^n$.
To show: If $z \in \mathbb{R}_{\ge 0}$ and $d(z, b) < \delta$ then $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Assume $z \in \mathbb{R}_{\ge 0}$ and $d(z, b) < \delta$.
To show: $d(z^{\frac{1}{n}}, b^{\frac{1}{n}}) < \epsilon$.
Let $a = b^{1/n}$ and $y = z^{1/n}$. Then

$$d(z^{1/n}, b^{1/n}) = d(y, a) < \frac{1}{na^{n-1}} d(y^n, a^n) = \frac{1}{na^{n-1}} d(z, b) < \frac{1}{na^{n-1}}\delta = \epsilon.$$

Since

$$d(y^n, a^n) = |y^n - a^n| = |(a+d)^n - a^n| = da^{n-1} \cdot \Big(\sum_{j=1}^{n} \frac{d^{j-1}}{a^{j-1}} \binom{n}{j}\Big) > da^{n-1}\binom{n}{1} = da^{n-1}n.$$

□

42.4 Topological properties of $\mathbb{R}_{\geq 0}$

Proposition 42.7. *(Topological properties of $\mathbb{R}_{\geq 0}$)*

(a) $\mathbb{R}_{\geq 0}$ *is a Hausdorff topological space.*

(b) $\mathbb{R}_{\geq 0}$ *is a complete metric space.*

(c) $\mathbb{R}_{\geq 0}$ *is locally compact.*

(d) $\mathbb{R}_{\geq 0}$ *is not compact.*

Proof. (Sketch)

(a) To show: If $x, y \in \mathbb{R}_{\geq 0}$ and $x \neq y$ then there exist open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.
Assume $x, y \in \mathbb{R}_{\geq 0}$ and $x \neq y$.
Let $\epsilon = \frac{1}{2}d(x, y)$ and let

$$U = \mathbb{R}_{(x-\epsilon,x+\epsilon)} \qquad \text{and} \qquad V = \mathbb{R}_{(y-\epsilon,y+\epsilon)}.$$

Then $x = x + 0 \in U$ and $y = y + 0 \in V$ and $U \cap V = \emptyset$.
So $\mathbb{R}_{\geq 0}$ is a Hausdorff topological space.

(b) To show: If $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$ then $(x_1, x_2, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.
To show: If $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$ then there exists $y \in \mathbb{R}_{\geq 0}$ such that $y = \lim\limits_{n\to\infty} x_n$.
Let $(x_1, x_2, \ldots)$ be a Cauchy sequence in $\mathbb{R}_{\geq 0}$.

$$\begin{aligned} x_1 &= z_1.d_{11}d_{12}d_{13}\ldots, \\ x_2 &= z_2.d_{21}d_{22}d_{23}\ldots, \\ x_3 &= z_3.d_{31}d_{32}d_{33}\ldots, \\ &\vdots \end{aligned}$$

To show: There exists $y \in \mathbb{R}_{\geq 0}$ such that $y = \lim\limits_{n\to\infty} x_n$.
For $k \in \mathbb{Z}_{\geq 0}$ let ℓ_k be such that if $m, n \in \mathbb{Z}_{\geq \ell_k}$ then $d(x_m, x_n) \leq 10^{-k}$.
Let $y = z.d_1d_2d_3\cdots$, where

$$z = z_{\ell_0}, \quad d_1 = d_{\ell_1 1}, \quad d_2 = d_{\ell_2 2}, \quad \ldots$$

To show: If $k \in \mathbb{Z}_{\geq 0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq N}$ then $d(x_m, y) < 10^{-k}$.
Assume $k \in \mathbb{Z}_{>0}$.
Let $N = \ell_{k+1}$.
To show: If $m \in \mathbb{Z}_{\geq \ell_{k+1}}$ then $d(x_m, y) < 10^{-k}$.
Assume $m \in \mathbb{Z}_{\geq \ell_{k+1}}$.
Then

$$d(x_m, y) \leq d(x_m, x_{\ell_{k+1}}) + d(x_{\ell_{k+1}}, y) < 10^{-(k+1)} + 10^{-(k+1)} < 10^{-k}.$$

So $\lim\limits_{k\in\infty} x_k = y$.
So Cauchy sequences in $\mathbb{R}_{\geq 0}$ converge.

So $\mathbb{R}_{\geq 0}$ is complete.

This proof is conceptual and easy but there is a little bit of fuzziness in this proof caused by the fact that the decimal expansion of an element of $\mathbb{R}_{\geq 0}$ is not uniquely determined, for example $0.999\ldots = 1.000\ldots$. To remove this fuzziness use equivalence classes of Cauchy sequences in $\widehat{\mathbb{Q}}_{\geq 0}$ as in the proof that the completion of a metric space is complete.

(c) To show: $\mathbb{R}_{\geq 0}$ is locally compact.

To show: (ca) $\mathbb{R}_{\geq 0}$ is Hausdorff.

(cb) If $x \in \mathbb{R}_{\geq 0}$ then there exists a neighborhood N of x such that N is cover compact.

(ca) By part (b), $\mathbb{R}_{\geq 0}$ is Hausdorff.

(cb) To show: If $x \in \mathbb{R}_{\geq 0}$ then there exists a neighborhood N of x such that N is cover compact.
Assume $x \in \mathbb{R}_{\geq 0}$.
Let $N = \overline{B_1(x)} = \{y \in \mathbb{R}_{\geq 0} \mid |y - x| \leq 1\}$.
Since $N \supseteq B_1(x)$ and $x \in B_1(x)$ then N is a neighborhood of x.
Since $N \subseteq B_2(x)$ then N is bounded.
Since N is closed and bounded then N is cover compact.

So $\mathbb{R}_{\geq 0}$ is locally compact.

(d) The sequence $(1, 2, 3, 4, \ldots)$ is a sequence in $\mathbb{R}_{\geq 0}$ that does not have a cluster point.

So $\mathbb{R}_{\geq 0}$ is not compact.

□

An *interval* in $\mathbb{R}_{\geq 0}$ is a set $A \subseteq \mathbb{R}_{\geq 0}$ such that

$$\text{if } x, y \in A \text{ and } z \in \mathbb{R}_{\geq 0} \text{ and } x < z < y \text{ then } z \in A.$$

Theorem 42.8. *Let $A \subseteq \mathbb{R}_{\geq 0}$.*

(a) A is connected if and only if A is an interval.

(b) A is compact if and only if A is closed and bounded.

Proof.

(a) $\Rightarrow$: Assume E is not an interval.
Let $x, y \in E$ and $z \in E^c$ with $x < z < y$.
Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$.
Then A and B are open sets of E and, since $x \in A$ and $y \in B$ then

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap B = \emptyset, \quad \text{and} \quad A \cup B = E.$$

So E is not connected.

(a) $\Leftarrow$: Assume E is an interval.
To show: E is connected.
Let $A \subseteq E$ and $B \subseteq E$ be open subsets of E such that

$$A \neq \emptyset, \quad B \neq \emptyset \quad \text{and} \quad A \cup B = E.$$

To show: $A \cap B \neq \emptyset$.
There exists $z \in A \cap B$.
Let $x_1, y_1 \in E$ with $x_1 \in A$ and $y_1 \in B$.

Switching A and B if necessary assume that $x_1 < y_1$.
Construct sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ by

$$x_{i+1} = \frac{x_i + y_i}{2} \quad \text{and} \quad y_{i+1} = y_i, \qquad \text{if} \quad \frac{x_i + y_i}{2} \in A,$$

$$x_{i+1} = x_i \quad \text{and} \quad y_{i+1} = \frac{x_i + y_i}{2}, \qquad \text{if} \quad \frac{x_i + y_i}{2} \in B.$$

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By induction, $x_i \in E$ and $y_i \in E$, and since E is an interval, $\frac{1}{2}(x_i + y_i) \in E$ so that

$$x_{i+1} \in E \qquad \text{and} \qquad y_{i+1} \in E.$$

Also

$$x_{i+1} \in A, \quad y_{i+1} \in B, \qquad x_i \le x_{i+1} < y_{i+1} \le y_i,$$

and

$$|x_{i+1} - y_{i+1}| \le \tfrac{1}{2}|x_i - y_i|, \qquad \text{so that} \qquad |x_{i+1} - y_{i+1}| \le \frac{1}{2^i}|x_1 - y_1|.$$

Theorem 17.10(a) says that increasing bounded sequences converge, and since the sequence $x_1, x_2, \ldots$ is increasing and bounded by y_1
then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.
Theorem 17.10(a) says that decreasing bounded sequences converge, and since the sequence $y_1, y_2, \ldots$ is decreasing and bounded by x_1
then $\lim_{n\to\infty} y_n$ exists in $\mathbb{R}$.
Since $\lim_{n\to\infty} |x_n - y_n| = 0$ then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.
Let

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Since $x_1 \le x_2 \le \cdots \le x_n < y_n \le y_{n-1} \le \cdots \le y_1$ for $n \in \mathbb{Z}_{>0}$ then

$$x_1 < z < y_1.$$

Since E is an interval, $z \in E$.
By the characterization of closure in metric spaces via limits (Theorem 13.6),

$$z = \lim_{n\to\infty} x_n \in \overline{A} \qquad \text{and} \qquad z = \lim_{n\to\infty} y_n \in \overline{B}.$$

Case 1: $z \in A$.

Since A is open then z is an interior point of A and there exists $\epsilon \in \mathbb{E}$ with $B_\epsilon(z) \subseteq A$.
Since $z \in \overline{B}$ then $B_\epsilon(z) \cap B \neq \emptyset$.
So $A \cap B \neq \emptyset$.

Case 2: $z \in B$.

Since B is open then z is an interior point of B and there exists $\epsilon \in \mathbb{E}$ with $B_\epsilon(z) \subseteq B$.
Since $z \in \overline{A}$ then $B_\epsilon(z) \cap A \neq \emptyset$.
So $A \cap B \neq \emptyset$.

So E is connected.

(b) By Theorem 4.1, E is compact if E is Cauchy compact and bounded, so

To show: (ba) If $E \subseteq \mathbb{R}$ is bounded then E is ball compact.

(bb) If $E \subseteq \mathbb{R}$ is closed then E is Cauchy compact.

(ba) Assume $E \subseteq \mathbb{R}$ is bounded.
To show: E is ball compact.
Since E is bounded there exists $x \in \mathbb{R}$ and $M \in \mathbb{R}_{>0}$ such that $E \subseteq (x - M, x + M)$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$ such that $E \subseteq B_\epsilon(x_1) \cup \cdots B_\epsilon(x_\ell)$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$ such that $E \subseteq B_\epsilon(x_1) \cup \cdots B_\epsilon(x_\ell)$.
Let $\ell \in \mathbb{Z}_{>0}$ such that $\ell \cdot \frac{\epsilon}{2} > 2M$. Let

$$x_1 = x - M, \quad x_2 = x_1 + \tfrac{\epsilon}{2}, \quad x_3 = x_2 + \tfrac{\epsilon}{2}, \ldots, x_\ell = x_1 + \ell\tfrac{\epsilon}{2}.$$

Then

$$\begin{aligned} E &\subseteq (x - M, x + M) \\ &\subseteq (x_1 - \tfrac{\epsilon}{2}, x_1 + \tfrac{\epsilon}{2}) \cup (x_2 - \tfrac{\epsilon}{2}, x_2 + \tfrac{\epsilon}{2}) \cup \cdots (x_\ell - \tfrac{\epsilon}{2}, x_\ell + \tfrac{\epsilon}{2}). \end{aligned}$$

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So E is ball compact.

(bb) Assume E is closed.
To show: E is Cauchy compact.
To show: E is complete.
To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in E then $(a_1, a_2, \ldots)$ converges in E.
Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in E.
Then $(a_1, a_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}$ is complete then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$.
To show: $\lim_{n\to\infty} a_n$ is an element of E.
Since E is closed,

$$E = \overline{E} = \big\{z \in \mathbb{R} \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } E \text{ with } z = \lim_{n\to\infty} a_n\big\}.$$

So $\lim_{n\to\infty} a_n \in \overline{E} = E$.
So $(a_1, a_2, \ldots)$ converges in E.
So E is complete.

So E is ball compact and Cauchy compact in the metric space $\mathbb{R}$.

So E is compact.

□