39 Tutorial 7: Compactness

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

39.0.1 Limit points are cluster points

Proposition 39.1. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. If $z \in X$ is a limit point of $(x_1, x_2, \ldots)$ then z is a cluster point of $(x_1, x_2, \ldots)$.*

39.0.2 Characterization of cluster points as limit points of subsequences

Proposition 39.2. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then $z \in X$ is a cluster point of $(x_1, x_2, \ldots)$ if and only if there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ such that z is a limit point of $(x_{n_1}, x_{n_2}, \ldots)$.*

39.0.3 Convergent sequences are Cauchy

Proposition 39.3. *Let (X, d) be a strict metric space. Let $(x_1, x_2, \ldots)$ be a convergent sequence in X. Then $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.*

39.0.4 Cluster points of Cauchy sequences are limit points

Proposition 39.4. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a Cauchy sequence in X. Let $z \in X$. Then z is a limit point of $(x_1, x_2, \ldots)$ if and only if z is a cluster point of $(x_1, x_2, \ldots)$.*

39.0.5 Sequences in ball compact spaces have Cauchy subsequences

Proposition 39.5. *Let (X, d) be a ball compact strict metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ which is Cauchy.*

39.0.6 Subspaces of ball compact spaces are ball compact

Proposition 39.6. *Let (X, d) be a ball compact metric space.*
If $A \subseteq X$ then (A, d) is ball compact.

39.0.7 cover compact $\Rightarrow$ ball compact

Proposition 39.7. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is ball compact.

39.0.8 Ball compact $\Rightarrow$ bounded

Proposition 39.8. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is ball compact compact then A is bounded.

39.0.9 cover compact $\Rightarrow$ sequentially compact

Proposition 39.9. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is sequentially compact.

39.0.10 sequentially compact $\Rightarrow$ Cauchy compact

Proposition 39.10. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is Cauchy compact.

39.0.11 Cauchy compact $\Rightarrow$ closed

Proposition 39.11. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is Cauchy compact then A is closed.

39.0.12 sequentially compact $\Rightarrow$ ball compact

Proposition 39.12. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is ball compact.

39.0.13 Ball compact + Cauchy compact $\Rightarrow$ sequentially compact

Proposition 39.13. *Let (X, d) be a complete metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq A$.*

if A is ball compact and Cauchy compact then A is sequentially compact.

39.0.14 Ball compact + Cauchy compact $\Rightarrow$ cover compact

Proposition 39.14. *Let (X, d) be a metric space and let $A \subseteq X$.*

If A is ball compact and Cauchy compact then A is cover compact.

40 Tutorial 7: Solutions

40.1 Limit points are cluster points

Proposition 40.1. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. If $z \in X$ is a limit point of $(x_1, x_2, \ldots)$ then z is a cluster point of $(x_1, x_2, \ldots)$.*

Proof. Let z be a limit point of $(x_1, x_2, \ldots)$.
To show: z is a cluster point of $(x_1, x_2, \ldots)$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $x_\ell \in B_\epsilon(z)$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that $x_\ell \in B_\epsilon(z)$.
Since z is a limit point of $(x_1, x_2, \ldots)$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $x_n \in B_\epsilon(z)$.
Let $\ell = N + 1$.
Then $x_\ell \in B_\epsilon(z)$.
So z is a cluster point of $(x_1, x_2, \ldots)$.

□

40.2 Characterization of cluster points as limit points of subsequences

Proposition 40.2. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then $z \in X$ is a cluster point of $(x_1, x_2, \ldots)$ if and only if there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ such that z is a limit point of $(x_{n_1}, x_{n_2}, \ldots)$.*

Proof.
$\Rightarrow$: Let $z \in X$ be a cluster point of $(x_1, x_2, \ldots)$.
For $k \in \mathbb{Z}_{>0}$ let n_k be such that $n_k > n_{k-1}$ and $x_{n_k} \in B_{10^{-k}}(z)$.
Then $x_{n_1}, x_{n_2}, \ldots)$ is a subsequence of $(x_1, x_2, \ldots)$ with z as a limit point.
$\Leftarrow$: $(x_{n_1}, x_{n_2}, \ldots)$ is a subsequence of $(x_1, x_2, \ldots)$ with z as a limit point.
If $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ with $x_{n_\ell} \in B_\varepsilon(z)$.
So z is a cluster point of $(x_1, x_2, \ldots)$. □

40.3 Convergent sequences are Cauchy

Proposition 40.3. *Let (X, d) be a strict metric space. Let $(x_1, x_2, \ldots)$ be a convergent sequence in X. Then $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.*

Proof. Let $(x_1, x_2, \ldots)$ be a convergent sequence in X.
Then there exists $z \in X$ such that $\lim_{k\to\infty} x_k = z$.
To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $(x_m, x_n) \in B_\epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Since $\lim_{k\to\infty} x_k = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq \ell}$ then $d(x_m, z) < \frac{\epsilon}{2}$.
To show: If $m, n \in \mathbb{Z}_{\geq \ell}$ then $(x_m, x_n) \in B_\epsilon$.
Assume $m, n \in \mathbb{Z}_{\geq \ell}$.
To show: $(x_m, x_n) \in B_\epsilon$.
To show: $d(x_m, x_n) < \epsilon$.

$$d(x_m, x_n) \leq d(x_m, z) + d(z, x_n) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

So $(x_m, x_n) \in B_\epsilon$.
So $(x_1, x_2, \ldots)$ is a Cauchy sequence. $\square$

40.4 Cluster points of Cauchy sequences are limit points

Proposition 40.4. *Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a Cauchy sequence in X. Let $z \in X$. Then z is a limit point of $(x_1, x_2, \ldots)$ if and only if z is a cluster point of $(x_1, x_2, \ldots)$.*

Proof. $\Rightarrow$: By Proposition 40.1, limit points are cluster points.
$\Leftarrow$: Let $(x_1, x_2, \ldots)$ be a Cauchy sequence in X and let z be a cluster point of $(x_1, x_2, \ldots)$.
To show: $z = \lim_{n\to\infty} x_n$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, z) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, z) < \epsilon$.
Let $N \in \mathbb{Z}_{>0}$ such that if $m, k \in \mathbb{Z}_{\geq N}$ then $d(x_m, x_k) < \frac{1}{2}\epsilon$; the value N exists since $x_1, x_2, \ldots$ is a Cauchy sequence in X.
Since z is a cluster point of $x_1, x_2, \ldots$ there exists $m \in \mathbb{Z}_{\geq N}$ such that $d(x_m, z) < \frac{1}{2}\epsilon$.
To show: If $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, z) < \epsilon$.
Assume $n \in \mathbb{Z}_{\geq N}$.
To show: $d(x_n, z) < \epsilon$.

$$d(x_n, z) \leq d(x_n, x_m) + d(x_m, z) \leq \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

So $z = \lim_{n\to\infty} x_n$. $\square$

40.5 Sequences in ball compact spaces have Cauchy subsequences

Proposition 40.5. *Let (X, d) be a ball compact strict metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ which is Cauchy.*

Proof.

Assume $(x_1, x_2, \ldots)$ is a sequence in X.
To show: There exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ which is Cauchy.
Since X is ball compact then $\{x_1, x_2, \ldots\}$ is ball compact.

Let $n_1 \in \mathbb{Z}_{>0}$ be minimal such that $B_1(x_{n_1})$ contains an infinite number of $(x_1, x_2, \ldots)$;

Let $n_2 \in \mathbb{Z}_{>n_1}$ be minimal such that $B_{\frac{1}{2}}(x_{n_2})$ contains an infinite number of $(x_1, x_2, \ldots) \cap B_1(x_{n_1})$;

Let $n_3 \in \mathbb{Z}_{>n_2}$ be minimal such that $B_{\frac{1}{3}}(x_{n_3})$ contains an infinite number of $(x_1, x_2, \ldots) \cap B_1(x_{n_1}) \cap B_{\frac{1}{2}}(x_{n_2})$;

etc.

To show: $(x_{n_1}, x_{n_2}, \ldots)$ is Cauchy.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_{n_r}, x_{n_s}) < \epsilon$.
Let $\ell \in \mathbb{Z}_{>0}$ such that $\dfrac{1}{\ell} < \dfrac{\epsilon}{2}$.
To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_{n_r}, x_{n_s}) < \epsilon$.
Assume $r, s \in \mathbb{Z}_{\geq \ell}$.

To show: $d(x_{n_r}, x_{n_s}) < \epsilon$.

$$d(x_{n_r}, x_{n_s}) \le d(x_{n_r}, x_{n_\ell}) + d(x_{n_\ell}, x_{n_s}) < \frac{1}{\ell} + \frac{1}{\ell} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $(x_{n_1}, x_{n_2}, \ldots)$ is Cauchy.

□

40.6 Subspaces of ball compact spaces are ball compact

Proposition 40.6. *Let (X, d) be a ball compact metric space.*
If $A \subseteq X$ then (A, d) is ball compact.

Proof. Assume X is ball compact and $A \subseteq X$.
To show: A is ball compact.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $B_\epsilon(a_1) \cup B_\epsilon(a_2) \cup \cdots \cup B_\epsilon(a_\ell) \supseteq A$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exist $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $B_\epsilon(a_1) \cup B_\epsilon(a_2) \cup \cdots \cup B_\epsilon(a_\ell) \supseteq A$.
Let $m \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in X$ such that $B_{\epsilon/2}(x_1) \cup \cdots \cup B_{\epsilon/2}(x_m) \supseteq X$.
Let ℓ be the number of $B_{\epsilon/2}(x_j)$ with $B_{\epsilon/2}(x_j) \cap A \neq \emptyset$.
For each $B_{\epsilon/2}(x_j)$ with $B_{\epsilon/2}(x_j) \cap A \neq \emptyset$ let $a_j \in B_{\epsilon/2}(x_j) \cap A \neq \emptyset$.
Then, since $B_\epsilon(a_j) \supseteq B_{\frac{\epsilon}{2}}(x_j)$,

$$B_\epsilon(a_1) \cup B_\epsilon(a_2) \cup \cdots \cup B_\epsilon(a_\ell) \supseteq (B_{\frac{\epsilon}{2}}(x_1) \cap A) \cup \cdots \cup (B_{\frac{\epsilon}{2}}(x_m) \cap A) \supseteq X \cap A = A,$$

So A is ball compact. □

40.7 cover compact ⇒ ball compact

Proposition 40.7. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is ball compact.

Proof. To show: If A is cover compact then A is ball compact.
Assume A is cover compact.
To show: A is ball compact.
To show: If $k \in \mathbb{Z}_{>0}$ and $\epsilon = 10^{-k}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.
Since A is cover compact and $\mathcal{S} = \{B_\epsilon(a) \mid a \in A\}$ is an open cover of A, there exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.
So A is ball compact.

□

40.8 Ball compact ⇒ bounded

Proposition 40.8. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is ball compact compact then A is bounded.

Proof. To show: If A is ball compact then A is bounded.
Assume A is ball compact.
To show: A is bounded.
To show: There exists $a \in A$ and $M \in \mathbb{R}_{>0}$ such that $A \subseteq B_M(a)$.
Since A is ball compact there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in A$ such that $A \subseteq B_1(x_1) \cup \cdots \cup B_1(x_\ell)$.
Let $a = x_1$ and let $M = 2 + \max\{d(x_1, x_1), d(x_1, x_2), d(x_1, x_3), \ldots, d(x_1, x_\ell)\}$.
Ta show: $A \subseteq B_M(a)$.
To show: If $x \in A$ then $d(x, a) < M$.
Assume $x \in A$.
To show: $d(x, a) < M$.
Let $j \in \{1, \ldots, \ell\}$ such that $x \in B_1(x_j)$.
Then

$$d(x, a) = d(x, x_1) \leq d(x, x_j) + d(x_j, x_1) \leq 1 + (M - 2) = M - 1 < M.$$

So $x \in B_M(a)$.
So $A \subseteq B_M(a)$.
So A is bounded.

□

40.9 cover compact ⇒ sequentially compact

Proposition 40.9. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is sequentially compact.

Proof. To show: If A is cover compact then A is sequentially compact.
To show: If A is not sequentially compact then A is not cover compact.

(Notes and References: See Lang's Real Analysis Chapter 2 §3 Proposition 3.7 for this proof)
Assume A is not sequentially compact.
Then there exists a sequence $a_1, a_2, \ldots$ in A with no cluster point in A.
Thus, if $z \in A$ then there exists $N \in \mathcal{N}(z)$ such that $\mathrm{Card}\{j \mid a_j \in N\}$ is finite.
To show: A is not cover compact.
To show: There exists an open cover $\mathcal{S}$ of A which does not have a finite subcover.
For $x \in A$ let V_x be an open set of A such that

$$x \in V_x \qquad \text{and} \qquad \mathrm{Card}\{j \in \mathbb{Z}_{>0} \mid a_j \in V_x\} \text{ is finite.}$$

The set V_x exists since x is not a cluster point of $(a_1, a_2, a_3, \ldots)$.
Then $\mathcal{S} = \{V_x \mid x \in A\}$ is an open cover of A.
To show: $\mathcal{S}$ does not contain a finite subcover of A.
Proof by contradiction.
Assume $\ell \in \mathbb{Z}_{>0}$ and $S_1, S_2, \ldots, S_\ell \in \mathcal{S}$ such that $S_1 \cup S_2 \cup \cdots \cup S_\ell \supseteq A$.
Let $k_j \in \mathbb{Z}_{>0}$ be such that if $n \in \mathbb{Z}_{\geq k_j}$ then $a_n \notin S_j$.
Let $k = \max\{k_1, k_2, \ldots, k_\ell\}$.
Then, if $n \in \mathbb{Z}_{>k}$ then $a_n \notin S_1 \cup \cdots \cup S_\ell$.
This is a contradiction to $S_1 \cup \cdots \cup S_\ell \supseteq A$.
So $\mathcal{S}$ has no finite subcover.
So A is not cover compact.

□

40.10 sequentially compact $\Rightarrow$ Cauchy compact

Proposition 40.10. *Let (X,d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is Cauchy compact.

Proof. To show: If A is sequentially compact then A is Cauchy compact.
Assume A is sequentially compact.
To show: A is Cauchy compact.
To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ has a limit point in A.
Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.
To show: There exists $a \in A$ such that $\lim_{n\to\infty} a_n = a$.
Let a be a cluster point of $(a_1, a_2, \ldots)$ in A, which exists since A is sequentially compact.
By Proposition 40.4, since $(a_1, a_2, \ldots)$ is Cauchy then the cluster point a is a limit point of $(a_1, a_2, \ldots)$.
So $a = \lim_{n\to\infty} a_n$.
So A is Cauchy compact. □

40.11 Cauchy compact $\Rightarrow$ closed

Proposition 40.11. *Let (X,d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is Cauchy compact then A is closed.

Proof. To show: If A is Cauchy compact then A is closed in X.
Assume $A \subseteq X$ and A is Cauchy compact.
To show: A is closed in X.
To show: If $(a_1, a_2, \ldots,)$ is a sequence in A and $(a_1, a_2, \ldots)$ converges in X then $\lim\limits_{k\to\infty} a_k \in A$.
Assume $(a_1, a_2, \ldots)$ is a sequence in A and $(a_1, a_2, \ldots)$ converges in X.
Since convergent sequences are Cauchy then $(a_1, a_2, \ldots)$ is a Cauchy sequence.
Since A is Cauchy compact and $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ converges in A.
Since limits in metric spaces are unique, $z = \lim\limits_{k\to\infty} a_k \in A$.
So A is closed in X. □

40.12 sequentially compact $\Rightarrow$ ball compact

Proposition 40.12. *Let (X,d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is ball compact.

Proof. To show: If A is sequentially compact then X is ball compact.
To show: Assume A is not ball compact.
To show: A is not sequentially compact.
To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with no cluster point in A.
Using that A is not ball compact, let $\epsilon \in \mathbb{R}_{>0}$ such that A is not covered by finitely many $B_\epsilon(x)$.
Let

$$a_1 \in A, \quad a_2 \in B_{\frac{\epsilon}{2}}(a_1)^c \cap A, \quad a_3 \in (B_{\frac{\epsilon}{2}}(a_1) \cup B_{\frac{\epsilon}{2}}(a_2))^c \cap A, \quad \ldots.$$

Then $(a_1, a_2, \ldots)$ has no cluster point, since every $B_{\frac{\epsilon}{2}}(x)$ contains at most one point of $(a_1, a_2, \ldots)$.
So A is not sequentially compact. □

40.13 Ball compact + Cauchy compact ⇒ sequentially compact

Proposition 40.13. *Let (X,d) be a complete metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq A$.*

$$\textit{if } A \textit{ is ball compact and Cauchy compact} \quad \textit{then} \quad A \textit{ is sequentially compact.}$$

Proof. Assume A is ball compact and Cauchy compact.
To show: A is sequentially compact.
To show: If $(a_1, a_2, \ldots)$ is a sequence in A then $(a_1, a_2, \ldots)$ has a cluster point in A.
Assume $(a_1, a_2, \ldots)$ is a sequence in A.
By Proposition 40.5, since A is ball compact there exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1, a_2, \ldots)$ such that $(a_{n_1}, a_{n_2}, \ldots)$ is Cauchy.
Since A is Cauchy compact $(a_{n_1}, a_{n_2}, \ldots)$ has a limit point in A.
Thus $(a_1, a_2, \ldots)$ has a cluster point A.
So A is sequentially compact.

□

40.14 Ball compact + Cauchy compact ⇒ cover compact

Proposition 40.14. *Let (X,d) be a metric space and let $A \subseteq X$.*

$$\textit{If } A \textit{ is ball compact and Cauchy compact} \quad \textit{then} \quad A \textit{ is cover compact.}$$

Proof. Assume A is ball compact.
To show: If A is Cauchy compact then A is cover compact.
To show: If A is not cover compact then A is not Cauchy compact.
Assume A is not cover compact.
Let $\mathcal{S}$ be an open cover with no finite subcover.
Let $a_1^{(1)}, \ldots, a_{\ell_1}^{(1)}$ be such that $B_{10^{-1}}(a_1^{(1)}) \cup \cdots \cup B_{10^{-1}}(a_{\ell_1}^{(1)}) \supseteq A$.
Let $a_{j_1}^{(1)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)})$ is not finitely covered by $\mathcal{S}$.
Let $a_1^{(2)}, \ldots, a_{\ell_2}^{(2)}$ be such that $B_{10^{-2}}(a_1^{(2)}) \cup \cdots \cup B_{10^{-2}}(a_{\ell_2}^{(2)}) \supseteq A \cap B_{10^{-1}}(a_{j_1}^{(1)})$.
Let $a_{j_2}^{(2)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)})$ is not finitely covered by $\mathcal{S}$.
Let $a_1^{(3)}, \ldots, a_{\ell_3}^{(3)}$ be such that $B_{10^{-3}}(a_1^{(3)}) \cup \cdots \cup B_{10^{-3}}(a_{\ell_3}^{(3)}) \supseteq A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)})$.
Let $a_{j_3}^{(3)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)}) \cap B_{10^{-3}}(a_{j_3}^{(3)})$ is not finitely covered by $\mathcal{S}$.
Continuing this process produces a sequence $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$ which is Cauchy (If $m, n \geq k+1$ then $d(a_{j_m}^{(m)}, a_{j_n}^{(n)}) \leq d(a_{j_m}^{(m)}, a_{j_{k+1}}^{(k+1)}) + d(a_{j_{k+1}}^{(k+1)}, a_{j_n}^{(n)}) \leq 10^{-(k+1)} + 10^{-(k+1)} \leq 10^{-k}$).
Let $z \in A$.
To show: z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$.
To show: There exists $\epsilon \in \mathbb{E}$ and $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_{j_n}^{(n)}, z) > \epsilon$.
Let $U \in \mathcal{S}$ such that $z \in U$.
Since U is open in X then there exists $k \in \mathbb{Z}_{>0}$ such that $B_{10^{-k}}(z) \subseteq U$.
Let $\epsilon = 10^{-k}$ and let $\ell = k$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_{j_n}^{(n)}, z) > \epsilon$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $B_{10^{-n}}(a_{j_n}^{(n)}) \not\subseteq B_{10^{-k}}(z)$ there exists $y \in B_{10^{-n}}(a_{j_n}^{(n)})$ such that $d(y,z) > 10^{-k}$.
Thus $d(a_{j_n}^{(n)}, z) \geq d(y,z) - d(a_{j_n}^{(n)}, y) > 10^{-k} - 10^{-n} > 10^{-k} = \epsilon$.

So z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$.
So A is not Cauchy compact. $\square$