43 Tutorial 10: Planning the execution of the exam

Below is Sample Exam 3. Make a plan for successfully executing the exam in a 3 hour window, with the goal of getting the bulk of the marks, and optimizing your marks overall. Discuss how you would go about it and what are the best strategies.

With your group, execute as much as you can during tutorial and make estimates of how many marks your team achieves.

Sample exam 3

Question 1. (10 Marks) Let (X, d) be a metric space.

(a) State the definition of sequential compactness.

(b) Suppose that X is sequentially compact and nonempty. Given $\epsilon > 0$ prove that there exists a finite set $x_1, \ldots, x_n \in X$ such that $\{B_\epsilon(x_i)\}_{i=1}^n$ covers X.

You must prove (b) directly from the definition of sequential compactness.

Question 2. (20 marks) Let X be a topological space, $\Delta = \{(x, x) \in X \times X \mid x \in X\}$. Prove

(a) X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) If X is Hausdorff and $f : Y \to X$ and $g : Y \to X$ are continuous maps and $A \subseteq Y$ is dense then $f = g$ if and only if $f(a) = g(a)$ for all $a \in A$.

Question 3. (20 marks) Let X be a locally compact Hausdorff space and Y, Z topological spaces. Let

$$\pi_Y : Y \times Z \to Y, \qquad \pi_Z : Y \times Z \to Z$$

be the projection maps. Prove that the function

$$\begin{array}{ccc} \mathrm{Cts}(X, Y \times Z) & \longrightarrow & \mathrm{Cts}(X, Y) \times \mathrm{Cts}(X, Z) \\ f & \longmapsto & (\pi_Y \circ f, \pi_Z \circ f) \end{array}$$

is a homeomorphism, with respect to the compact-open topology. You may assume the universal property of the product, and the adjunction property for the compact-open topology (including continuity of evaluation maps).

Question 4. (20 marks) Let $(V, \| \, \|)$ be a normed space over a field of scalars $\mathbb{F}$ (which recall denotes either $\mathbb{R}$ or $\mathbb{C}$).

(a) Prove that $\| \, \| : V \to \mathbb{F}$ is uniformly continuous.

Prove that V is a topological vector space by proving

(b) The addition $V \times V \to V$ is continous.

(c) The scalar multiplication $\mathbb{F} \times V \to V$ is continuous.

You may prove continuity using either the product topology or the product metric.

Question 5. (20 marks) Let $(V, \| \, \|)$ be a normed space over a field of scalars $\mathbb{F}$ and let $V^\vee$ denote the space of continuous linear maps $V \to \mathbb{F}$ with the operator norm. You may assume that this is a normed space. Prove that this space is *complete*, as follows:

(a) Given a Cauchy sequence $(T_n)_{n=0}^{\infty}$ in $V^\vee$ with respect to the operator norm, construct a candidate limit T as a function $T\colon V \to \mathbb{F}$.

(b) Prove that your candidate T is linear.

(b) Prove that your candidate T is bounded.

(d) Prove that $T_n \to T$ in the operator norm as $n \to \infty$.

Question 6. (20 marks) Let $(H, \langle\ \rangle)$ be a Hilbert space over $\mathbb{C}$.

(a) State the Cauchy-Schwartz inequality.

(b) Prove that for any $h \in H$ the function $\langle \cdot, h\rangle\colon H \to \mathbb{C}$ is continuous.

(c) Prove that if $\{u_i\}_{i\in I}$ is a set of vectors in H which span a vector subspace $U \subseteq H$ with the property that U is dense in H, then $H = 0$ if and only if $\langle u_i, h\rangle = 0$ for all $i \in I$.

(d) Given that $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ span a dense subspace of $H = L^2(S^1, \mathbb{C})$ prove that for every $f \in H$

$$f = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{2\pi} \langle f, e^{in\theta}\rangle e^{in\theta}.$$

You may assume that the series on the right hand side converges.