6 Uniform spaces

Let X be a set. The set of (ordered) pairs of elements of X is

$$X \times X = \{(x_1, x_2) \mid x_1, x_2 \in X\}. \qquad \text{The } \textit{diagonal} \text{ is} \quad \Delta(X) = \{(x, x) \mid x \in X\},$$

a subset of $X \times X$. For $E \subseteq X \times X$ let

$$\sigma(E) = \{(y, x) \in X \times X \mid (x, y) \in D\}, \qquad \text{and}$$
$$E \times_X E = \{(x, y) \in X \times X \mid \text{there exists } z \in X \text{ such that } (x, z) \in E \text{ and } (z, y) \in E\}.$$

A *uniformity on* X is a collection $\mathcal{E}$ of subsets of $X \times X$ such that

(a) (diagonal condition) If $E \in \mathcal{E}$ then $\Delta(X) \subseteq E$,

(b) (upper ideal) If $E \in \mathcal{E}$ and $D \subseteq X \times X$ and $D \supseteq E$ then $D \in \mathcal{E}$,

(c) (finite intersection) If $\ell \in \mathbb{Z}_{>0}$ and $E_1, E_2, \ldots, E_\ell \in \mathcal{E}$ then $E_1 \cap E_2 \cap \cdots \cap E_\ell \in \mathcal{E}$,

(d) (symmetry condition) If $E \in \mathcal{E}$ then $\sigma(E) \in \mathcal{E}$,

(e) (triangle condition) If $E \in \mathcal{E}$ then there exists $D \in \mathcal{E}$ such that $D \times_X D \subseteq E$.

A *uniform space* is a set X with a uniformity $\mathcal{E}$ on X. An *fatdiagonal*, or *entourage*, is a set in $\mathcal{E}$.

6.1 Uniform spaces can be made into topological spaces

Let $(X, \mathcal{E})$ be a uniform space.

Let $E \in \mathcal{E}$ and $x \in X$. The *E-neighborhood of* x is

$$B_E(x) = \{y \in X \mid (x, y) \in E\}.$$

Let $x \in X$. The *neighborhood filter* of x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } E \in \mathcal{X} \text{ such that } N \supseteq B_E(x)\}.$$

The *uniform space topology* on X is the topology

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } E \in \mathcal{E} \text{ such that } B_E(x) \subseteq U\}.$$

Proposition 6.1. *Let $(X, \mathcal{E})$ be a uniform space. Let $x \in X$ and let $\mathcal{N}(x)$ be the neighborhood filter of x for the uniform space topology. Then*

$$\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}.$$

6.2 The categories of topological and uniform spaces

Continuous functions are for comparing topological spaces.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A *continuous function from X to Y* is a function $f: X \to Y$ such that

$$\text{if } V \in \mathcal{T}_Y \quad \text{then} \quad f^{-1}(V) \in \mathcal{T}_X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

An *isomorphism of topological spaces*, or *homeomorphism*, is a continuous function $f: X \to Y$ such that the inverse function $f^{-1}: Y \to X$ exists and is continuous.

Uniformly continuous functions are for comparing uniform spaces.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces. A *uniformly continuous function from X to Y* is a function $f: X \to Y$ such that

$$\text{if } E \in \mathcal{E}_Y \quad \text{then} \quad (f \times f)^{-1}(E) \in \mathcal{E}_X.$$

An *isomorphism of uniform spaces* is a uniformly continuous function $f: X \to Y$ such that the inverse function $f^{-1}: Y \to X$ exists and is uniformly continuous.

The following proposition is the key point for establishing that $\mathcal{T}op$, topological spaces with continuous functions, and $\mathcal{U}nif$, uniform spaces with uniformly continuous functions, are categories and that the uniform space topology provides a functor $\mathcal{U}nif \to \mathcal{T}op$.

Proposition 6.2.

(a) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then

$$g \circ f: X \to Z \quad \text{is a continuous function.}$$

(b) Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ and $(Z, \mathcal{E}_Z)$ be uniform spaces and let $f: X \to Y$ and $g: Y \to Z$ be uniformly continuous functions. Then

$$g \circ f: X \to Z \quad \text{is a uniformly continuous function.}$$

(c) Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces. Let $\mathcal{T}_X$ be the uniform space topology on $(X, \mathcal{E}_X)$ and let $\mathcal{T}_Y$ be the uniform space topology on $(Y, \mathcal{E}_Y)$.

$$\text{If } f: X \to Y \text{ is uniformly continuous} \quad \text{then} \quad f: X \to Y \text{ is continuous.}$$

6.3 Metric spaces can be made into uniform spaces

A tolerance is a number of decimal places of accuracy to achieve in a measurement. The *set of tolerances* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

Let (X, d) be a metric space.

- Let $x \in X$ and $\epsilon \in \mathbb{E}$. The *open ball of radius ϵ at x* is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

- Let $\epsilon \in \mathbb{E}$. The *diagonal of width ϵ*, or *ϵ-diagonal*, is

$$B_\epsilon = \{(y, x) \in X \times X \mid d(x, y) < \epsilon\}.$$

Let $x \in X$. The *neighborhood filter of* x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } \epsilon \in \mathbb{E} \text{ such that } N \supseteq B_\epsilon(x)\}.$$

The *metric space topology on* X is

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The *metric space uniformity on* X is

$$\mathcal{E} = \{\text{subsets of } X \times X \text{ which contain an } \epsilon\text{-diagonal}\}.$$

More precisely, $E \subseteq X \times X$ is a *fatdiagonal in* X if and only if

$$\text{there exists } \epsilon \in \mathbb{E} \text{ such that} \quad E \supseteq B_\epsilon.$$

6.4 epsilon-delta characterizations of continuity and uniform continuity

The set

$$\mathbb{E} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\} \qquad \text{is the } \textit{accuracy set.}$$

Specifying an element of $\mathbb{E}$ specifies the desired number of decimal places of accuracy.

Proposition 6.3. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$ be a function.*

(a) The function $f\colon X \to Y$ is continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ and } a \in X \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } x \in X \textit{ and } d_X(a,x) < \delta \textit{ then } d_Y(f(a), f(x)) < \epsilon.\end{aligned}$$

(b) The function $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } a, x \in X \textit{ and } d_X(a,x) < \delta \textit{ then } d_Y(f(a), f(x)) < \epsilon.\end{aligned}$$

Proposition 6.4. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces and let $f\colon X \to Y$ be a function.*

(a) The function $f\colon X \to Y$ is continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } E \in \mathcal{E}_Y \textit{ and } a \in X \textit{ then there exists } D \in \mathcal{E}_X \textit{ such that}\\ &\qquad \textit{if } x \in X \textit{ and } (a,x) \in D \textit{ then } (f(a), f(x)) \in E.\end{aligned}$$

(b) The function $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{aligned}&\textit{if } E \in \mathcal{E}_Y \textit{ then there exists } D \in \mathcal{E}_X \textit{ such that}\\ &\qquad \textit{if } a, x \in X \textit{ and } (a,x) \in D \textit{ then } (f(a), f(x)) \in E.\end{aligned}$$

6.5 Uniform spaces come from metric spaces

If $E \subseteq X \times X$ then

$$E \bowtie E = \{(x,y) \in X \times X \mid \text{there exists } z \in X \text{ such that } (x,z) \in E \text{ and } (z,y) \in E\}.$$

Let $(X, \mathcal{E})$ be a uniform space. For each $E \in \mathcal{E}$ fix $E_1, E_2, E_3, \ldots \in \mathcal{E}$ given by

$$E_1 = E \cap \sigma(E) \quad \text{and} \quad E_{k+1} = D_{k+1} \cap \sigma(D_{k+1})$$

where $D_{k+1} \in \mathcal{E}$ is chosen such that $D_{k+1} \bowtie D_{k+1} \subseteq E_k$. Fix $U_1, U_2, \ldots \in \mathcal{E}$ given by

$$U_1 = E_1, \qquad \text{and} \qquad U_{k+1} = F_{k+1} \cap \sigma(F_{k+1})$$

where $F_{k+1} \in \mathcal{E}$ is chosen such that $F_{k+1} \bowtie F_{k+1} \bowtie F_{k+1} \subseteq (U_k \cap E_k)$. Define $g\colon X \times X \to \mathbb{R}_{\geq 0}$ by

$$g(x,y) = \begin{cases} 1, & \text{if } (x,y) \notin U_1, \\ 2^{-k}, & \text{if } (x,y) \in U_1, (x,y) \in U_2, \ldots, (x,y) \in U_k \text{ and } (x,y) \notin U_{k+1}, \\ 0, & \text{if } (x,y) \in U_k \text{ for } k \in \mathbb{Z}_{>0}, \end{cases} \tag{6.1}$$

and define $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$d(x,y) = \inf\{g(x,z_1) + g(z_1,z_2) + \cdots + g(z_{p-1},y) \mid p \in \mathbb{Z}_{>0},\ z_1, \ldots, z_p \in X,\ z_p = y\}. \tag{6.2}$$

Let

$$\begin{aligned} \mathcal{X}_E &= \{D \subseteq X \times X \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ such that } D \supseteq E_k\}, \\ \mathcal{X}_U &= \{D \subseteq X \times X \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ such that } D \supseteq U_k\} \quad \text{and} \\ \mathcal{X}_d &= \{D \subseteq X \times X \mid \text{there exists } \epsilon \in \mathbb{E} \text{ such that } D \supseteq B_\epsilon\}, \end{aligned} \tag{6.3}$$

so that $\mathcal{X}_d$ is the metric space topology on (X,d).

The following proposition tells us that every uniform space $(X, \mathcal{E})$ can be obtained as a supremum of uniformities that come from metrics.

Proposition 6.5. *Let $(X, \mathcal{E})$ be a uniform space.*

(a) Let $E \in \mathcal{E}$ and let $d, \mathcal{X}_E, \mathcal{X}_U$ and $\mathcal{X}_d$ be as defined in (6.2) and (6.3). Then

$$\mathcal{X}_E \text{ is a uniformity on } X, \quad d \text{ is a metric on } X, \quad \text{and} \quad \mathcal{X}_E = \mathcal{X}_U = \mathcal{X}_d.$$

(b) $\mathcal{E} = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.

6.6 Notes and References

6.6.1 Math is not broken: uniform continuity

When I was an undergraduate, a graduate student and for the first 15 years of my career as a professional mathematician, the difference between continuous and uniformly continuous functions was terrifying to me. This was compounded with the definitions of uniform convergence and pointwise convergence which, all together, left me in a serious muddle. At some point, perhaps the second time I was teaching this topic out of the Baby Rudin book [BRu], my "proof machine" skills had finally gotten strong enough that I was able to get all the "for all" and "for each" out of the definitions, put them only in an "if-then-there exists" form and actually look and see what the logical differences were:

A function $f\colon X \to Y$ is *continuous* if f satisfies:

if $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that
if $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$.

A function $f\colon X \to Y$ is *uniformly continuous* if f satisfies:

if $\epsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that
if $x \in X$ and $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{array}{l}\text{if } x \in X \text{ and } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \\ \quad \text{if } n \in \mathbb{Z}_{\geq \ell} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{array}$$

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{array}{l}\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \\ \quad \text{if } x \in X \text{ and } n \in \mathbb{Z}_{\geq \ell} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{array}$$

The first few times I met this subject I was not proficient enough at logical manipulation to even parse the huge numbers of quantifiers in a definition like the definition of a continuous function. I had just been too glazed over to realize that pointwise convergence and uniform convergence are about *sequences* of functions so I should not have been mixing those up with continuity and uniform continuity at all.

I was thrilled when I learned from Bourbaki that there are *uniform spaces* and **uniformly continuous functions are for comparing uniform spaces in the same way that continuous functions are for comparing topological spaces**.

6.6.2 The bridge

With uniform spaces, this part of math has a beautiful structure, where the "fatdiagonals" of a uniform space are for keeping track of which pairs of points are within ϵ of each other in the same way that for metric spaces the "open sets" are trying to keeping track of the points in the ϵ-ball around a central point. Cauchy sequences are a natural construct in the uniform space context, and compactness and completeness now have their proper homes in the world of topological spaces and uniform spaces, respectively.

The definition of uniform spaces in Section 11.2 follows [Bou, Top. Ch. II]. It is structured to model and highlight the analogies to topological spaces, and to provide a bridge between topological spaces and metric spaces. It is helpful to remember that the elements of a uniformity are called "entourages" or "fatdiagonals", in the same way that the elements of a topology are called "open sets". **The category of uniform spaces is the natural home for uniformly continuous functions, Cauchy sequences and completion.**

The relation between metric spaces and uniform spaces becomes vivid through Propositions 6.3 and 6.4 which give epsilon-delta characterizations of continuity and uniform continuity for metric spaces and uniform spaces. Via these results uniform spaces form a perfect bridge between topological spaces and metric spaces.

6.6.3 Uniform spaces and the category of metric spaces

Proposition 6.5 says that that if the collection of strict metric spaces is enlarged to include metrics that take value ∞ and to include 'limits' (supremums) of metric spaces then one naturally obtains the category of uniform spaces. Hence, in the same way that one discovers the real numbers by filling in the holes in the set of rational numbers (by filling in the limit points), one discovers that the category of uniform spaces is what is obtained by filling in the holes in the set of metric spaces (by filling in the limit spaces).

The uniform spaces with the uniformly continuous functions form a category since a composition of uniformly continuous functions is uniformly continuous. The definition of the uniform space topology then gives a functor from the category of uniform spaces to the category of topological spaces.

6.7 Some proofs

6.7.1 The neighborhood filter of a uniform space

Proposition 6.6. *Let $(X, \mathcal{E})$ be a uniform space. Let $x \in X$ and let $\mathcal{N}(x)$ be the neighborhood filter of x for the uniform space topology. Then*

$$\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}.$$

Proof. To show: $\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}$.
By definition $\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } E \in \mathcal{E} \text{ such that } N \supseteq B_E(x)\}$.
To show: (a) $\mathcal{N}(x) \supseteq \{B_E(x) \mid E \in \mathcal{E}\}$.
(b) $\mathcal{N}(x) \subseteq \{B_E(x) \mid E \in \mathcal{E}\}$.

(a) This is direct from the definition of $\mathcal{N}(x)$.
(b) To show: If $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{E}$ such that $N = B_W(x)$.
Assume $N \in \mathcal{N}(x)$.
Then there exists $E \in \mathcal{E}$ with $N \supseteq B_E(x)$.
To show: There exists $W \in \mathcal{E}$ such that $N = B_W(x)$.
Let $W = \{(y, x) \mid y \in N\}$.
If $(y, x) \in E$ then $y \in B_E(x) \subseteq N$ and so $(y, x) \in W$.
Thus $W \supseteq E$.
Since $E \in \mathcal{E}$ and $W \subseteq X \times X$ and $W \supseteq E$ then $W \in \mathcal{E}$.
To show: $N = B_W(x)$.
To show: (a) $N \subseteq B_W(x)$.
(b) $B_W(x) \subseteq N$.

(a) Assume $n \in N$.
Then $(n, x) \in W$ and $n \in B_W(x)$.
So $N \subseteq B_W(x)$.
(b) Assume $y \in B_W(x)$.
Then $(y, x) \in W$.
Thus, by the definition of W, $y \in N$.
So $B_W(x) \subseteq N$.

So $N = B_W(x)$.

So $\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{E}\}$. □

6.8 Uniformly continuous functions

Proposition 6.7.

(a) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then

$$g \circ f : X \to Z \quad \textit{is a continuous function.}$$

(b) Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ and $(Z, \mathcal{E}_Z)$ be uniform spaces and let $f: X \to Y$ and $g: Y \to Z$ be uniformly continuous functions. Then

$$g \circ f : X \to Z \quad \textit{is a uniformly continuous function.}$$

Proof. (a) To show: If $V \in \mathcal{T}_Z$ then $(g \circ f)^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Z$.
Since g is continuous then $g^{-1}(V) \in \mathcal{T}_Y$.
Since f is continuous then $f^{-1}(g^{-1}(V)) \in \mathcal{T}_X$.
So

$$\begin{aligned} f^{-1}(g^{-1}(V)) &= \{x \in X \mid f(x) \in g^{-1}(V)\} \\ &= \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid (g \circ f)(x)) \in V\} \\ &= (g \circ f)^{-1}(V) \text{ is an element of } \mathcal{T}_X. \end{aligned}$$

So $g \circ f$ is continuous.

(b) To show: If $V \in \mathcal{E}_Z$ then $((g \circ f) \times (g \circ f))^{-1}(V) \in \mathcal{E}_X$.
Assume $V \in \mathcal{E}_Z$.
Since g is uniformly continuous then $(g \times g)^{-1}(V) \in \mathcal{E}_Y$.
Since f is uniformly continuous then $(f \times f)^{-1}((g \times g)^{-1}(V)) \in \mathcal{E}_X$.
So

$$\begin{aligned} (f \times f)^{-1}((g \times g)^{-1}(V)) &= \{(x_1, x_2) \in X \times X \mid f(x_1), f(x_2)) \in (g \times g)^{-1}(V)\} \\ &= \{(x_1, x_2) \in X \times X \mid (g(f(x_1)), g(f(x_2))) \in V\} \\ &= \{(x_1, x_2) \in X \times X \mid ((g \circ f)(x_1), (g \circ f)(x_2)) \in V\} \\ &= \{(x_1, x_2) \in X \times X \mid ((g \circ f) \times (g \circ f))(x_1, x_2)) \in V\} \\ &= ((g \circ f) \times (g \circ f))^{-1}(V) \text{ is an element of } \mathcal{E}_X. \end{aligned}$$

So $g \circ f$ is uniformly continuous.

□

6.9 Uniformly continuous functions are continuous

Proposition 6.8. *Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces. Let $\mathcal{T}_X$ be the uniform space topology on $(X, \mathcal{X}_X)$ and let $\mathcal{T}_Y$ be the uniform space topology on $(Y, \mathcal{X}_Y)$.*

If $f\colon X \to Y$ is uniformly continuous then $f\colon X \to Y$ is continuous.

Proof. Assume $f\colon X \to Y$ is uniformly continuous.

To show: $f\colon X \to Y$ is continuous.
To show: If $a \in A$ then $f\colon X \to Y$ is continuous at a.
Assume $a \in X$.
To show: f is continuous at a.

To show: If $V \in \mathcal{N}(f(a))$ then $f^{-1}(V) \in \mathcal{N}(a)$.
Assume $V \in \mathcal{N}(f(a))$.
To show: $f^{-1}(V) \in \mathcal{N}(a)$.

To show: There exists $D \in \mathcal{E}_X$ such that $f^{-1}(V) \supseteq B_D(a)$.
Since $V \in \mathcal{N}(f(a))$ there exists $C \in \mathcal{E}_Y$ such that $V \supseteq B_C(f(a))$.
Let $D = (f \times f)^{-1}(C)$.

To show: $f^{-1}(V) \supseteq B_D(a)$.
To show: If $y \in B_D(a)$ then $y \in f^{-1}(V)$.
Assume $y \in B_D(a)$.
Then $(a, y) \in D$.
So $(a, y) \in (f \times f)^{-1}(C)$.
So $(f(a), f(y)) \in C$.
So $f(y) \in B_C(f(a))$.
So $f(y) \in V$.
So $y \in f^{-1}(V)$.
So $f^{-1}(V) \supseteq B_D(a)$.
So $f^{-1}(V) \in \mathcal{N}(a)$.
So f is continuous at a.

So f is continuous. □

6.10 epsilon-delta characterization of continuity and uniform continuity

Proposition 6.9. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be a function.*

(a) The function $f: X \to Y$ is continuous if and only if f satisfies

if $\epsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that
if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

(b) The function $f: X \to Y$ is uniformly continuous if and only if f satisfies

if $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that
if $a, x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

Proof. (a) $\Rightarrow$: Assume $f: X \to Y$ is continuous.

To show: If $\epsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$ and $a \in X$.
To show: There exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Since f is continuous and $B_\epsilon(f(a))$ is open in Y then $f^{-1}(B_\epsilon(f(a)))$ is open in X.
Using that $f^{-1}(B_\epsilon(f(a)))$ is open in X and $a \in f^{-1}(B_\epsilon(f(a)))$,

$$\text{let } \delta \in \mathbb{E} \quad \text{such that} \quad B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a))).$$

To show: If $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $x \in X$ and $d_X(a, x) < \delta$.
To show: $d_Y(f(a), f(x)) < \epsilon$.
Since $x \in B_\delta(a)$ and $B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a)))$ then

$$f(x) \in B_\epsilon(f(a)).$$

So $d_Y(f(a), f(x)) < \epsilon$.

(a) $\Leftarrow$: Assume: If $\epsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

To show: $f: X \to Y$ is continuous.

Let $\mathcal{T}_X$ be the metric space topology for (X, d_X).
Let $\mathcal{T}_Y$ be the metric space topology for (Y, d_Y).

To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
Then $f(a) \in V$ and, since V is open in Y,

$$\text{there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(f(a)) \subseteq V.$$

To show: a is an interior point of $f^{-1}(V)$.
To show: There exists $\gamma \in \mathbb{E}$ such that $B_\gamma(a) \subseteq f^{-1}(V)$.
We know there exists $\delta \in \mathbb{E}$ such that 'if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$'.
Let $\gamma = \delta$.
To show: $B_\gamma(a) \subseteq f^{-1}(V)$.
Since δ satisfies 'if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$',

$$\text{then} \quad f(B_\gamma(a)) = f(B_\delta(a)) \subseteq B_\epsilon(f(a)).$$

Since $B_\epsilon(f(a)) \subseteq V$ then $B_\gamma(a) \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So $f: X \to Y$ is continuous.

(b) $\Rightarrow$: Assume $f: X \to Y$ is uniformly continuous.

To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

Let $\mathcal{E}_X$ be the metric space uniformity for (X, d_X).
Let $\mathcal{E}_Y$ be the metric space uniformity for (Y, d_Y).

Assume $\epsilon \in \mathbb{E}$.
To show: There exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Since f is uniformly continuous and $B_\epsilon \in \mathcal{E}_Y$ then $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$.
Since $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$, then

$$\text{there exists } \gamma \in \mathbb{E} \quad \text{such that} \quad B_\gamma \subseteq (f \times f)^{-1}(B_\epsilon).$$

Let $\delta = \gamma$.
To show: If $a, x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.
Assume $a, x \in X$ and $d_X(a, x) < \delta$.
To show: $d_Y(f(a), f(x)) < \epsilon$.
Since $d_X(a, x) < \delta = \gamma$

$$\text{then} \quad (a, x) \in B_\gamma \subseteq (f \times f)^{-1}(B_\epsilon).$$

So $(f(a), f(x)) \in B_\epsilon$.
So $d_Y(f(a), f(x)) < \epsilon$.

(b) $\Leftarrow$: Assume: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $a, x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$.

To show: $f\colon X \to Y$ is uniformly continuous.

Let $\mathcal{E}_X$ be the metric space uniformity for (X, d_X).
Let $\mathcal{E}_Y$ be the metric space uniformity for (Y, d_Y).

To show: If $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Assume $E \in \mathcal{E}_Y$.
To show: $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
To show: There exists $\gamma \in \mathbb{E}$ such that $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since $E \in \mathcal{E}_Y$

$$\text{there exists } \epsilon \in \mathbb{E} \quad \text{such that} \quad B_\epsilon \subseteq E.$$

So there exists $\delta \in \mathbb{E}$ such that

$$\text{if } a, x \in X \text{ and } d_X(a, x) < \delta \text{ then } d_Y(f(a), f(x)) < \epsilon. \qquad (*)$$

Let $\gamma = \delta$.
To show: $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since δ satisfies (*), then if $(a, x) \in B_\delta = B_\gamma$ then $(f(a), f(x)) \in B_\epsilon$.
So $(f \times f)(B_\gamma) \subseteq B_\epsilon \subseteq E$.
So $B_\gamma \subseteq (f \times f)^{-1}(E)$.
So $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
So $f\colon X \to Y$ is uniformly continuous.

□

Proposition 6.10. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces and let $f\colon X \to Y$ be a function.*

(a) The function $f\colon X \to Y$ is continuous if and only if f satisfies

$$\text{if } E \in \mathcal{E}_Y \text{ and } a \in X \text{ then there exists } D \in \mathcal{E}_X \text{ such that}$$
$$\text{if } x \in X \text{ and } (a, x) \in D \text{ then } (f(a), f(x)) \in E.$$

(b) The function $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

$$\text{if } E \in \mathcal{E}_Y \text{ then there exists } D \in \mathcal{E}_X \text{ such that}$$
$$\text{if } a, x \in X \text{ and } (a, x) \in D \text{ then } (f(a), f(x)) \in E.$$

Proof. (a) $\Rightarrow$: Assume $f\colon X \to Y$ is continuous.

To show: If $E \in \mathcal{E}_Y$ and $a \in X$ then there exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $E \in \mathcal{E}_Y$ and $a \in X$.
To show: There exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Since f is continuous and $B_E(f(a))$ is open in Y then $f^{-1}(B_E(f(a)))$ is open in X.
Using that $f^{-1}(B_E(f(a)))$ is open in X and $a \in f^{-1}(B_E(f(a)))$,

$$\text{let } D \in \mathcal{E}_X \quad \text{such that} \quad B_D(a) \subseteq f^{-1}(B_E(f(a))).$$

To show: If $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $x \in X$ and $(a, x) \in D$.

To show: $(f(a), f(x)) \in E$.
Since $x \in B_D(a)$ and $B_D(a) \subseteq f^{-1}(B_E(f(a)))$ then $f(x) \in B_E(f(a))$.
So $(f(a), f(x)) \in E$.

(a) $\Leftarrow$: Assume: If $E \in \mathcal{E}_Y$ and $a \in X$ then there exists $D \in \mathcal{E}_X$ such that if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.

To show: $f: X \to Y$ is continuous.

Let $\mathcal{T}_X$ be the uniform space topology for $(X, \mathcal{E}_X)$.
Let $\mathcal{T}_Y$ be the uniform space topology for $(Y, \mathcal{E}_Y)$.

To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
Then $f(a) \in V$ and, since V is open in Y,

$$\text{there exists } E \in \mathcal{E}_Y \text{ such that } B_E(f(a)) \subseteq V.$$

To show: a is an interior point of $f^{-1}(V)$.
To show: There exists $G \in \mathcal{E}_X$ such that $B_G(a) \subseteq f^{-1}(V)$.
We know there exists $D \in \mathcal{E}_X$ such that 'if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$'.
Let $G = D$.
To show: $B_G(a) \subseteq f^{-1}(V)$.
Since D satisfies 'if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$',

$$\text{then} \quad f(B_G(a)) = f(B_D(a)) \subseteq B_E(f(a)).$$

Since $B_E(f(a)) \subseteq V$ then $B_G(a) \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So $f: X \to Y$ is continuous.
So $f: X \to Y$ is continuous.

(b) $\Rightarrow$: Assume $f: X \to Y$ is uniformly continuous.

To show: If $E \in \mathcal{E}_Y$ then there exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $E \in \mathcal{E}_Y$.
To show: There exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Since f is uniformly continuous and $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Since $(f \times f)^{-1}(B_\epsilon) \in \mathcal{E}_X$, then

$$\text{there exists } G \in \mathcal{E}_X \quad \text{such that} \quad G \subseteq (f \times f)^{-1}(E).$$

Let $D = G$.
To show: If $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.
Assume $a, x \in X$ and $(a, x) \in D$.
To show: $(f(a), f(x)) \in E$.

Since $(a, x) \in D = G$

$$\text{then} \quad (a, x) \in G \subseteq (f \times f)^{-1}(E).$$

So $(f(a), f(x)) \in E$.
So $d_Y(f(a), f(x)) \in E$.

(b) $\Leftarrow$: Assume: If $E \in \mathcal{E}_Y$ then there exists $D \in \mathcal{E}_X$ such that if $a, x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.

To show: $f\colon X \to Y$ is uniformly continuous.
To show: If $E \in \mathcal{E}_Y$ then $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
Assume $E \in \mathcal{E}_Y$.
To show: $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
To show: There exists $G \in \mathcal{E}_X$ such that $G \subseteq (f \times f)^{-1}(E)$.
Since $E \in \mathcal{E}_Y$ there exists $D \in \mathcal{E}_X$ such that

$$\text{if } a, x \in X \text{ and } (a, x) \in D \text{ then } (f(a), f(x)) \in E. \qquad (*)$$

Let $G = D$.
To show: $B_\gamma \subseteq (f \times f)^{-1}(E)$.
Since D satisfies (*), then if $(a, x) \in D = G$ then $(f(a), f(x)) \in E$.
So $(f \times f)(G \subseteq E$.
So $G \subseteq (f \times f)^{-1}(E)$.
So $(f \times f)^{-1}(E) \in \mathcal{E}_X$.
So $f\colon X \to Y$ is uniformly continuous.

□

6.10.1 Making metric spaces into uniform spaces

Proposition 6.11. *Let (X, d) be a metric space and let $\mathcal{E}_d$ be the metric space uniformity, as defined in (??). Then $(X, \mathcal{E}_d)$ is a uniform space.*

Proof. To show: $\mathcal{E}$ is a uniformity.
To show: (a) If $E \in \mathcal{E}$ then $E \supseteq \Delta(X)$.
(b) If $E \in \mathcal{E}$ and $D \subseteq X \times X$ and $D \supseteq E$ then $D \in \mathcal{E}$.
(c) If $E \in \mathcal{E}$ then $\sigma(E) \in \mathcal{E}$.
(d) If $E_1, \ldots, E_\ell \in \mathcal{E}$ then $E_1 \cap \cdots \cap E_\ell \in \mathcal{E}$.
(e) If $E \in \mathcal{E}$ then there exists $D \in \mathcal{E}$ such that $D \bowtie D \subseteq E$.

(a) Assume $E \in \mathcal{E}$.
To show: $E \supseteq \Delta(X)$.
Since there exists $\epsilon \in \mathbb{E}$ with $E \supseteq B_\epsilon$ and

$$B_\epsilon = \{(x, y) \in X \times X \mid d(x, y) < \epsilon\} \supseteq \{(x, y) \in X \times X \mid d(x, y) = 0\} \supseteq \Delta(X),$$

then $E \supseteq B_\epsilon \supseteq \Delta(X)$.

(b) Assume $E \in \mathcal{E}$ and $D \subseteq X \times X$ and $D \supseteq E$.
To show: $D \in \mathcal{E}$.
Since $E \in \mathcal{E}$ there exists $\epsilon \in \mathbb{E}$ such that $E \supseteq B_\epsilon$.
So $D \supseteq E \supseteq B_\epsilon$.
So there exists $\epsilon \in \mathbb{E}$ such that $D \supseteq B_\epsilon$.

So $D \in \mathcal{E}$.

(c) Assume $E \in \mathcal{E}$.
To show: $\sigma(E) \in \mathcal{E}$.
Let $\epsilon \in \mathbb{E}$ such that $V \supseteq B_\epsilon$.
To show: $\sigma(E) \supseteq B_\epsilon$.

$$\sigma(V) = \{(y,x) \mid (x,y) \in E\} \supseteq \{(y,x) \mid (x,y) \in B_\epsilon\} = \{(y,x) \mid d(x,y) < \epsilon\} = B_\epsilon.$$

So there exists $\epsilon \in \mathbb{E}$ such that $\sigma(E) \supseteq B_\epsilon$.
So $\sigma(E) \in \mathcal{E}$.

(d) Assume $\ell \in \mathbb{Z}_{>0}$ and $E_1, \ldots, E_\ell \in \mathcal{E}$.
To show: $E_1 \cap \cdots \cap E_\ell \in \mathcal{E}$.
To show: THere exists $\epsilon \in \mathbb{E}$ such that $E_1 \cap \cdots \cap E_\ell \supseteq B_\epsilon$.
Let

$$\begin{aligned} &\epsilon_1 \in \mathbb{E} \text{ such that } E_1 \supseteq B_{\epsilon_1}, \\ &\epsilon_2 \in \mathbb{E} \text{ such that } E_1 \supseteq B_{\epsilon_2}, \ \ldots \\ &\epsilon_\ell \in \mathbb{E} \text{ such that } E_1 \supseteq B_{\epsilon_\ell}. \end{aligned}$$

Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_\ell\}$.
To show: $E_1 \cap \cdots \cap E_\ell \supseteq B_\epsilon$.

$$E_1 \cap \cdots \cap E_\ell \supseteq B_{\epsilon_1} \cap \cdots \cap B_{\epsilon_\ell} = B_\epsilon.$$

So $E_1 \cap \cdots \cap E_\ell \in \mathcal{E}$.

(e) Assume $E \in \mathcal{E}$.
To show: There exists $D \in \mathcal{E}$ such that $D \bowtie D \subseteq E$.
Let $\epsilon \in \mathbb{E}$ such that $E \supseteq B_\epsilon$.
Let $D = B_{\epsilon/2}$.
To show: $D \bowtie D \subseteq E$.
Since $D \supseteq B_{\epsilon/2}$ then $D \in \mathcal{E}$ and

$$D \bowtie D = B_{\epsilon/2} \bowtie B_{\epsilon/2} \subseteq B_\epsilon,$$

since $d(x,y) \le d(x,z) + d(z,y) < \epsilon$ if $(x,y) \in B_{\epsilon/2} \bowtie B_{\epsilon/2}$.
So $D \bowtie D \subseteq B_\epsilon \subseteq E$.

□

6.10.2 Uniform spaces come from metric spaces

Proposition 6.12. *Let $(X, \mathcal{E})$ be a uniform space.*

(a) Let $E \in \mathcal{E}$ and let $d, \mathcal{X}_E, \mathcal{X}_U$ and $\mathcal{X}_d$ be as defined in (6.2) and (6.3). Then

$$\mathcal{X}_E \text{ is a uniformity on } X, \quad d \text{ is a metric on } X, \quad \text{and} \quad \mathcal{X}_E = \mathcal{X}_U = \mathcal{X}_d.$$

(b) $\mathcal{E} = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.

Proof. Note that $E_1, E_2, \ldots \in \mathcal{E}$ and $U_1, U_2, \ldots \in \mathcal{E}$ such that

$$E_1 \subseteq E, \qquad \sigma(E_n) = E_n, \quad \text{and} \quad E_{n+1} \subseteq E_{n+1} \bowtie E_{n+1} \subseteq E_n, \quad \text{and} \tag{6.4}$$

$$U_1 \subseteq E_1, \quad \sigma(U_n) = U_n, \quad \text{and} \quad U_{n+1} \subseteq U_{n+1} \bowtie U_{n+1} \bowtie U_{n+1} \subseteq (U_n \cap E_n). \tag{6.5}$$

To show: (a) d is a metric.
(b) If $x, y \in X$ then $d(x,y) \le g(x,y)$.
(c) If $x, y \in X$ then $d(x,y) \ge \frac{1}{2}g(x,y)$.
(d) $\mathcal{X}_U = \mathcal{X}_d$.
(e) $\mathcal{X}_E = \mathcal{X}_U$.
(f) $\mathcal{E} = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.

(a) To show: d is a metric.
To show: (aa) If $x \in X$ then $d(x,x) = 0$.
(ab) If $x, y \in X$ then $d(x,y) = d(y,x)$.
(ac) If $x, y, z \in X$ then $d(x,y) \le d(x,z) + d(z,y)$.

(aa) Assume $x \in X$.
To show: $d(x,x) = 0$.

$$d(x,x) = \inf\{g(x,z_1) + \cdots + g(z_{p-1},x) \mid p \in \mathbb{Z}_{>0}\} \le g(x,x) = 0,$$

since $(x,x) \in U_n$ for $n \in \mathbb{Z}_{>0}$.
(ab) Assume $x, y \in X$.
If $a, b \in X$ then $g(a,b) = g(b,a)$ since $\sigma(U_n) = U_n$, and so

$$\begin{aligned} d(x,y) &= \inf\{g(x,z_1) + \cdots + g(z_{p-1},y) \mid p \in \mathbb{Z}_{>0}\} \\ &= \inf\{g(y,z_{p-1}) + \cdots + g(z_1,x) \mid p \in \mathbb{Z}_{>0}\} = d(y,x). \end{aligned}$$

(ac) Assume $x, y, z \in X$.
To show: $d(x,y) \le d(x,z) + d(z,y)$.

$$\begin{aligned} d(x,y) &= \inf\{g(x,z_1) + \cdots + g(z_{p-1},y) \mid p \in \mathbb{Z}_{>0}\} \\ &\le \inf\{g(x,v_1) + \cdots + g(v_{k-1},z) + g(z,w_1) + \cdots + g(w_{r-1},y) \mid k, r \in \mathbb{Z}_{>0}\} \\ &= \inf\{g(x,v_1) + \cdots + g(v_{k-1},z) \mid k \in \mathbb{Z}_{>0}\} + \inf\{g(z,w_1) + \cdots + g(w_{r-1},y) \mid r \in \mathbb{Z}_{>0}\} \\ &= d(x,z) + d(z,y). \end{aligned}$$

(b) To show: $d(x,y) \le g(x,y)$.
Since $g(x,y) \in \{g(x,z_1) + \cdots + g(z_{p-1},y) \mid p \in \mathbb{Z}_{>0}\}$ then $d(x,y) \le g(x,y)$.
(c) To show: $d(x,y) \ge \frac{1}{2}g(x,y)$.
To show: If $p \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_{p-1} \in X$ then $\big(g(x,z_1) + \cdots g(z_{p-1},y)\big) \ge \frac{1}{2}g(x,y)$.
By induction on p.

The base case $p = 1$: $g(z_0,z_1) = g(x,y) \ge \frac{1}{2}g(x,y)$.
Induction step:
Assume $p \in \mathbb{Z}_{>1}$.
Case A: If $\big(g(x,z_1) + \cdots g(z_{p-1},z_p)\big) \ge \frac{1}{2}$ then

$$g(x,z_1) + \cdots g(z_{p-1},z_p) \ge \tfrac{1}{2} \ge \tfrac{1}{2} \cdot 1 \ge \tfrac{1}{2}g(x,y).$$

Case B: Assume $\big(g(x, z_1) + \cdots g(z_{p-1}, z_p)\big) < \frac{1}{2}$.
Write

$$\big(g(z_0, z_1) + \cdots + g(z_{h-1}, z_h)\big) + g(z_h, z_{h+1}) + \big(g(z_{h+1}, z_{h+2}) + \cdots + g(z_{p-1}, z_p)\big)$$

with $h \in \{1, \ldots, p-1\}$ (CAN p BE 2????) such that

$$\begin{array}{lll} (g(z_0, z_1) + \cdots + g(z_{h-1}, z_h)) & \le \frac{1}{2}\big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big), & \text{and} \\ (g(z_0, z_1) + \cdots + g(z_{h-1}, z_h)) + g(z_h, z_{h+1}) & > \frac{1}{2}\big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big) & \text{so that} \\ \big(g(z_{h+1}, z_{h+2}) + \cdots + g(z_{p-1}, z_p)\big) & \le \frac{1}{2}\big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big), & \end{array}$$

By induction,

$$\begin{array}{rl} \frac{1}{2}g(x, z_h) \le g(z_0, z_1) + \cdots + g(z_{h-1}, z_h) & \text{and} \\ \frac{1}{2}g(z_{h+1}, y) \le g(z_{h+1}, z_{h+2}) + \cdots + g(z_{p-1}, z_p). & \end{array}$$

So

$$\begin{array}{rl} g(z_h, z_{h+1}) &\le \big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big), \\ \frac{1}{2}g(x, z_h) &\le g(z_0, z_1) + \cdots + g(z_{h-1}, z_h) \le \frac{1}{2}(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)) \\ \frac{1}{2}g(z_{h+1}, y) &\le \big(g(z_{h+1}, z_{h+2}) + \cdots + g(z_{p-1}, z_p)\big) \le \frac{1}{2}(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)) \end{array}$$

Let $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_{>0}$ such that $g(x, z_h) = 2^{-\ell_1}$, $g(z_h, z_{h+1}) = 2^{-\ell_2}$, $g(z_{h+1}, y) = 2^{-\ell_3}$.
Then

$$\begin{aligned} 2^{-\ell_1} &= g(z_h, z_{h+1}) \le (g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)), \\ 2^{-\ell_2} &= g(x, z_h) \le (g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)) \\ 2^{-\ell_3} &= g(z_{h+1}, y) \le (g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)) \end{aligned}$$

Let $k \in \mathbb{Z}_{>0}$ be minimal such that $2^{-k} \le \big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big) < \frac{1}{2}$.
Since $2^{-k} \le \big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big) < \frac{1}{2}$ then

$$\ell_1 \ge k > 1 \quad \text{and} \quad \ell_2 \ge k > 1 \quad \text{and} \quad \ell_3 \ge k > 1.$$

So $(x, z_h) \in U_k$, $(z_{h+1}, y) \in U_k$ and $(z_h, z_{h+1}) \in U_k$.
Using the third property in (6.5), $(x, y) \in U_k \bowtie U_k \bowtie U_k \subseteq U_{k-1}$.
So $g(x, y) \le 2^{-(k-1)} = 2 \cdot 2^{-k} \le 2\big(g(z_0, z_1) + \cdots + g(z_{p-1}, z_p)\big)$.
So $g(x, y) \le 2d(x, y)$.

(d) To show: $\mathcal{X}_E = \mathcal{X}_U$.
To show: (da) $\mathcal{X}_E \subseteq \mathcal{X}_U$.
(db) $\mathcal{X}_U \subseteq \mathcal{X}_E$.

(da) Assume $D \in \mathcal{X}_E$.
Let $n \in \mathbb{Z}_{>0}$ such that $D \supseteq E_n$.
Using the thrid property in (6.5), Since $U_{n+1} \subseteq U_{n+1} \bowtie U_{n+1} \bowtie U_{n+1} \subseteq U_n \cap E_n \subseteq E_n \subseteq D$.
So $D \in \mathcal{X}_U$.
So $\mathcal{X}_E \subseteq \mathcal{X}_U$.

(db) Assume $D \in \mathcal{X}_U$.
Let $n \in \mathbb{Z}_{>0}$ such that $U_n \subseteq D$.

By the ??? property in (6.5), $U_n \subseteq E_{n-1}$ if $n > 1$.
By definition of U_k, then $U_k \in \mathcal{X}_E$. REALLY????
So $\mathcal{X}_U \subseteq \mathcal{X}_E$.

So $\mathcal{X}_U = \mathcal{X}_E$.

(e) To show: $\mathcal{X}_d = \mathcal{X}_U$.
To show: (ea) $\mathcal{X}_d \subseteq \mathcal{X}_U$.
(eb) $\mathcal{X}_U \subseteq \mathcal{X}_d$.

(ea) If $(x, y) \in U_k$ then $d(x, y) \le g(x, y) \le 2^{-k}$.
So $(x, y) \in B_{2^{-k}}$.
So $U_k \subseteq B_{2^{-k}}$.
So $B_{2^{-k}} \in \mathcal{X}_U$.
So $\mathcal{X}_d \subseteq \mathcal{X}_U$.

(eb) Using (c), if $(x, y) \in B_{2^{-k}}$ then $g(x, y) \le 2d(x, y) \le 2^{-(k-1)}$.
So $(x, y) \in U_{k-1}$.
So $B_{2^{-k}} \subseteq U_{k-1}$.
So $U_{k-1} \in \mathcal{X}_d$.
So $\mathcal{X}_U \subseteq \mathcal{X}_d$.

So $\mathcal{X}_U = \mathcal{X}_d$.

(f) To show: $\mathcal{E} = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.
To show: (fa) $\mathcal{E}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.
(fb) If $\mathcal{Y}$ is a uniformity on X and $\mathcal{Y}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{E}\}$ then $\mathcal{Y} \supseteq \mathcal{E}$.

(fa) To show: $\mathcal{E} \supseteq \mathcal{X}_E$.
To show: If $D \in \mathcal{X}_E$ then $D \in \mathcal{E}$.
Assume $D \in \mathcal{X}_E$.
To show: There exists $k \in \mathbb{Z}_{>0}$ such that $D \supseteq E_k$.
Let $k \in \mathbb{Z}_{>0}$ such that $D \supseteq E_k$.
Using the upper ideal condition on $\mathcal{E}$,
since $E_k \in \mathcal{E}$ and $D \subseteq X \times X$ and $D \supseteq E_k$ then $D \in \mathcal{E}$.
So $\mathcal{X}_E \subseteq \mathcal{E}$.

(fb) To show: If $\mathcal{Y}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{E}\}$ then $\mathcal{Y} \supseteq \mathcal{E}$.
Assume $\mathcal{Y}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.
To show: $\mathcal{Y} \supseteq \mathcal{E}$.
We know: If $E \in \mathcal{E}$ then $\mathcal{Y} \supseteq \mathcal{X}_E$.
To show: If $E \in \mathcal{E}$ then $E \in \mathcal{Y}$.
Assume $E \in \mathcal{E}$.
To show: $E \subseteq \mathcal{Y}$.
Since $E_1 = E$ and $E_1 \in \mathcal{X}_E$ then $E \in \mathcal{X}_E$.
Since $\mathcal{X}_E \subseteq \mathcal{Y}$ then $E \in \mathcal{Y}$.
So $\mathcal{E} \subseteq \mathcal{Y}$.

So $\mathcal{E}$ is a least upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.
So $\mathcal{E} = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}\}$.

□