5.3.1 Inverse by determinants

Theorem 5.7. *(Inverse by determinants) Let* $A \in M_n(\mathbb{F})$ *such that* $\det(A)$ *is invertible. Then the inverse of* A *is the matrix* A^{-1} *given by*

$$A^{-1}(i,j) = \frac{1}{\det(A)}(-1)^{i+j}\det(A^{(j;i)}).$$

where $A^{(i;j)}$ *is the matrix* A *with the* i*th and the* j*th column removed.*

5.3.2 Cramer's rule

Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_n(\mathbb{F})$ and assume that A is invertible.

$$\text{Let}\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad\text{and}\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \text{be elements of } \mathbb{F}^n \text{ with} \quad Ax = b.$$

For $i \in \{1, \ldots, n\}$ let

$$b \xrightarrow{i} A \qquad \text{be the matrix } A \text{ except with } i\text{th column replaced by } b.$$

Then

$$x_1 = \frac{\det(b \xrightarrow{1} A)}{\det(A)}, \quad x_2 = \frac{\det(b \xrightarrow{2} A)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(b \xrightarrow{n} A)}{\det(A)}.$$

5.4 The Cayley-Hamilton theorem

Let $\mathbb{F}[x]$ be the algebra of polynomials in the variable x. Let $A \in M_n(\mathbb{F})$. Define

$$\begin{array}{cccc} \mathrm{ev}_A\colon & \mathbb{F}[x] & \longrightarrow & M_n(\mathbb{F}) \\ & c_0 + c_1 x + \cdots + c_r x^r & \longmapsto & c_0 + c_1 A + \cdots c_r A^r \end{array}$$

Theorem 5.8. *(Cayley-Hamilton) Let* $\ker(\mathrm{ev}_A) = \{p(x) \in \mathbb{F}[x] \mid \mathrm{ev}_A(p(x)) = 0.\}$ *Then*

$$\det(A - x) \in \ker(\mathrm{ev}_A).$$