4 Matrix groups: generators and relations

Let $n \in \mathbb{Z}_{>0}$ and let E_{ij} be the matrix which has 1 in the (i,j) entry and all other entries 0.

4.1 Diagonal matrices T_n

Let $\mathbb{F}^\times = \{d \in \mathbb{F} \mid d \neq 0\}$. Let $n \in \mathbb{Z}_{>0}$. Use the notation

$$\mathrm{diag}(d_1, \ldots, d_n) = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad \text{for } d_1, \ldots, d_n \in \mathbb{F}^\times,$$

so that d_i is the diagonal entry in the ith row and ith column and all other entries of $\mathrm{diag}(d_1, \ldots, d_n)$ are 0.

- An $n \times n$ *diagonal matrix* is an $n \times n$ matrix A such that if $i, j \in \{1, \ldots, n\}$ and $i \neq j$ then $A(i,j) = 0$.
- The *elementary diagonal matrices* are the matrices

$$h_i(d) = 1 + (-1+d)E_{ii} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & d & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \qquad \text{for } d \in \mathbb{F}^\times.$$

- The *diagonal torus* is

$$T_n = \{\mathrm{diag}(d_1, \ldots, d_n) \mid d_1, \ldots, d_n \in \mathbb{F}^\times\} = \{h_1(d_1) \cdots h_n(d_n) \mid d_1, \ldots, d_n \in \mathbb{F}^\times\}.$$

Proposition 4.1. *The diagonal torus T_n is presented by generators*

$$h_i(d), \qquad \text{for } i \in \{1, \ldots, n\} \text{ and } d \in \mathbb{F}^\times,$$

with relations

$$h_i(d_1)h_i(d_2) = h_i(d_1 d_2) \qquad \text{and} \qquad h_i(d_1)h_j(d_2) = h_j(d_2)h_i(d_1), \tag{Tnrels}$$

for $i, j \in \{1, \ldots, n\}$ and $d_1, d_2 \in \mathbb{F}^\times$.

Proof. Since $\mathrm{diag}(d_1, \ldots, d_n) = h_1(d_1) \cdots h_n(d_n)$ each element of T_n can be written in terms of the generators $h_i(d)$. Since

$$\begin{aligned} \mathrm{diag}(d_1, \ldots, d_n)\mathrm{diag}(e_1, \ldots, e_n) &= h_1(d_1) \cdots h_n(d_n) h_1(e_1) \cdots h_n(e_n) \\ &= h_1(d_1)h_1(e_1) \cdots h_n(d_n)h_n(e_n) \\ &= h_1(d_1e_1) \cdots h_n(d_ne_n) = \mathrm{diag}(d_1e_1, \ldots, d_ne_n), \end{aligned}$$

where the second equality follows from the second relation in (Tnrels) and the third equality follows from the first relation in (Tnrels). Thus the relations in (Tnrels) determine the multiplication of diagonal matrices. So the group T_n is determined by the generators $h_i(d)$ and the relations in (Tnrels). □