4.4 Invertible matrices $GL_n(\mathbb{F})$

Let $n \in \mathbb{Z}_{>0}$, let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$ and let

$$E_{ij} \text{ be the } n \times n \text{ matrix with 1 in the } (i,j) \text{ entry and 0 elsewhere.}$$

- An $n \times n$ *invertible matrix* is an $n \times n$ matrix $A \in M_n(\mathbb{F})$ such that

$$\text{there exists } A^{-1} \in M_n(\mathbb{F}) \quad \text{such that} \quad A^{-1}A = 1 \text{ and } AA^{-1} = 1.$$

- The *general linear group* is

$$GL_n(\mathbb{F}) = \{n \times n \text{ invertible matrices with entries in } \mathbb{F}\}.$$

The invetrible elements of the field $\mathbb{F}$ are the elements of

$$\mathbb{F}^\times = \{d \in \mathbb{F} \mid d \neq 0\} = \{1 \times 1 \text{ invertible matrices with entries in } \mathbb{F}\} = GL_1(\mathbb{F}).$$

- The *elementary matrices in* $GL_n(\mathbb{F})$ are the matrices

$$s_{ij} = 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}, \qquad \text{for } i,j \in \{1,\ldots,n\} \text{ with } i \neq j,$$

$$x_{ij}(c) = 1 + cE_{ij}, \qquad \text{for } i,j \in \{1,\ldots,n\} \text{ with } i \neq j \text{ and } c \in \mathbb{F},$$
$$h_i(d) = 1 + (d-1)E_{ii}, \qquad \text{for } i \in \{1,\ldots,n\} \text{ and } d \in GL_1(\mathbb{F}).$$

- The *row reducers* are $y_i(c) = x_{i,i+1}(c)s_{i,i+1}$ for $i \in \{1,\ldots,n-1\}$ and $c \in \mathbb{F}$.

$$y_i(c) = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& c & 1 && \\ &&& 1 & 0 && \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \qquad \text{and} \qquad y_i(c)^{-1} = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& 0 & 1 && \\ &&& 1 & -c && \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix}$$

Theorem 4.4. *The group $GL_n(\mathbb{F})$ is presented by generators*

$$y_i(c),\quad h_j(d),\quad x_{k\ell}(c),\qquad \text{for}\qquad \begin{array}{l} c\in\mathbb{F}, d_1,\ldots,d_n\in\mathbb{F}^\times, \\ i\in\{1,\ldots,n-1\}, j\in\{1,\ldots,n\} \\ k,\ell\in\{1,\ldots,n\} \text{ with } k<\ell. \end{array}$$

with the following relations:

- *The* reflection relation *is*

$$y_i(c_1)y_i(c_2) = \begin{cases} y_i(c_1+c_2^{-1})h_i(c_2)h_{i+1}(-c_2^{-1})x_{i,i+1}(c_2^{-1}), & \text{if } c_2\neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2=0. \end{cases} \tag{4.3}$$

- *The* building relation *is*

$$y_i(c_1)y_{i+1}(c_2)y_i(c_3) = y_{i+1}(c_3)y_i(c_1c_3+c_2)y_{i+1}(c_1). \tag{4.4}$$

- *The* x-interchange relations *are*

$$\begin{array}{ll} x_{ij}(c_1)x_{ij}(c_2) = x_{ij}(c_1+c_2), & \\ x_{ij}(c_1)x_{ik}(c_2) = x_{ik}(c_2)x_{ij}(c_1), & x_{ik}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ij}(c_1)x_{ik}(c_1c_2), & x_{jk}(c_1)x_{ij}(c_2) = x_{ij}(c_2)x_{jk}(c_1)x_{ik}(-c_1c_2), \end{array}$$

  where $i<j<k$.

- *Letting $h(d_1,\ldots,d_n)=h_1(d_1)\cdots h_n(d_n)$, the* h-past-y relation *is*

$$h(d_1,\ldots d_n)y_i(c) = y_i(cd_id_{i+1}^{-1})h(d_1,\ldots,d_{i-1},d_{i+1},d_i,d_{i+2},\ldots,d_n). \tag{4.5}$$

- *Letting $h(d_1,\ldots,d_n)=h_1(d_1)\cdots h_n(d_n)$, the* h-past-x relation *is*

$$h(d_1,\ldots,d_n)x_{ij}(c) = x_{ij}(cd_id_j^{-1})h(d_1,\ldots,d_n). \tag{4.6}$$

- *The* x-past-y relations *are*

$$\begin{array}{c} x_{i,i+1}(c_1)y_i(c_2) = y_i(c_1+c_2)x_{i,i+1}(0), \\ x_{ik}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1c_2)x_{i,k+1}(c_1), \qquad x_{i,k+1}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)y_i(c_2) = y_i(c_2)x_{i+1,j}(c_1), \qquad x_{i+1,j}(c_1)y_i(c_2) = y_i(c_2)x_{ij}(c_1)x_{i+1,j}(-c_1c_2), \end{array} \tag{4.7}$$

  where $i<k$ and $i+1<j$.