3 Solving systems of linear equations

3.1 Invertible matrices

A matrix $P \in M_n(\mathbb{F})$ is *invertible* if there exists a matrix $P^{-1} \in M_n(\mathbb{F})$ such that

$$PP^{-1} = 1 \qquad \text{and} \qquad P^{-1}P = 1.$$

The *general linear group* is

$$GL_n(\mathbb{F}) = \{P \in M_n(\mathbb{F}) \mid P \text{ is invertible}\}.$$

Proposition 3.1. *If $P, Q \in GL_n(\mathbb{F})$ then*

$$(PQ)^{-1} = Q^{-1}P^{-1}.$$

3.2 Kernels and images

Let $\mathbb{F}^n = M_{n\times 1}(\mathbb{F})$. A *subspace of* $\mathbb{F}^n$ is a subset $V \subseteq \mathbb{F}^n$ such that

(a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,
(b) if $v \in V$ and $c \in \mathbb{F}$ then $cv \in V$.

Let $A \in M_{m\times n}(\mathbb{F})$. Define

$$\ker(A) = \{v \in \mathbb{F}^n \mid Av = 0\} \qquad \text{and} \qquad \mathrm{im}(A) = \{Av \mid v \in \mathbb{F}^n\}.$$

Proposition 3.2. *Let $A \in M_{m\times n}(\mathbb{F})$. Then $\ker(A)$ is a subspace of $\mathbb{F}^n$ and $\mathrm{im}(A)$ is a subspace of $\mathbb{F}^m$.*

Proposition 3.3. *Let $\mathbb{F}$ be a field and let $A \in M_{m\times n}(\mathbb{F})$. Let $P^{-1} \in GL_m(\mathbb{F})$ and $Q^{-1} \in GL_n(\mathbb{F})$. Then*

$$\ker(P^{-1}AQ^{-1}) = Q\ker(A) \qquad \textit{and} \qquad \mathrm{im}(P^{-1}AQ^{-1}) = P^{-1}\,\mathrm{im}(A).$$

Let $1_r \in M_{m\times n}(\mathbb{F})$ be given by $1_r = E_{11} + \cdots + E_{rr}$. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ let $e_i \in \mathbb{F}^n$ and $f_j \in \mathbb{F}^m$ be given by $e_i = E_{i1}$ and $f_j = E_{j1}$. Then

$$\{e_1, \ldots, e_n\} \text{ is a basis of } \mathbb{F}^n \qquad \text{and} \qquad \{f_1, \ldots, f_m\} \text{ is a basis of } \mathbb{F}^m.$$

Then

$$\{e_{r+1}, \ldots, e_n\} \text{ is a basis of } \ker(1_r) \qquad \text{and} \qquad \mathrm{im}(1_r) = \mathrm{span}\{f_1, \ldots, f_r\}. \qquad \text{(kerimbasis)}$$

Proposition 3.4. *Let $A \in M_{m\times n}(\mathbb{F})$. Let $r \in \{1, \ldots, \min(m, n)\}$ and $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ such that $A = P\mathbf{1}_rQ$. Then*

$$\ker(A) = Q^{-1}\ker(1_r) \qquad \textit{and} \qquad \mathrm{im}(A) = P\,\mathrm{im}(1_r).$$

Proposition 3.5. *Let $A \in M_{m\times n}(\mathbb{F})$. Then*

$$\dim(\mathrm{im}(A)) = (\textit{number of columns of } A) - \dim(\ker(A)).$$

Remark 3.6. The terms rank and nullity should be deprecated as it is more accurate and more instructive to use the phrases "dimension of the image" and "dimension of the kernel",

$$\mathrm{nullity}(A) = \dim(\ker(A)) \qquad \text{and} \qquad \mathrm{rank}(A) = \dim(\mathrm{im}(A)).$$