3.4 Some proofs

3.4.1 The inverse of PQ

Proposition 3.10. *If $P, Q \in GL_n(\mathbb{F})$ then*

$$(PQ)^{-1} = Q^{-1}P^{-1}.$$

Proof. Assume $P, Q \in GL_n(\mathbb{F})$.
To show: $(PQ)^{-1} = Q^{-1}P^{-1}$.
To show: (a) $(PQ)(Q^{-1}P^{-1}) = 1$.
(b) $(Q^{-1}P^{-1})(PQ) = 1$.
(a) Using associativity of matrix multiplication,

$$(PQ)(Q^{-1}P^{-1}) = P(QQ^{-1})P^{-1} = P \cdot 1 \cdot P^{-1} = PP^{-1} = 1$$

(b) Using associativity of matrix multiplication,

$$(Q^{-1}P^{-1})(PQ) = Q^{-1}(P^{-1}P)Q = Q^{-1} \cdot 1 \cdot Q = Q^{-1}Q = 1.$$

So $(PQ)^{-1} = Q^{-1}P^{-1}$. □

3.4.2 $\ker(A)$ and $\mathrm{im}(A)$ are subpaces

Proposition 3.11. *Let $A \in M_{m\times n}(\mathbb{F})$. Then $\ker(A)$ is a subspace of $\mathbb{F}^n$ and $\mathrm{im}(A)$ is a subspace of $\mathbb{F}^m$.*

Proof. (aa) Assume $v_1, v_2 \in \ker(A)$. Then

$$A(v_1 + v_2) = Av_1 + Av_2 = 0 + 0 = 0.$$

So $v_1 + v_2 \in \ker(A)$. (ab) Assume $v \in \ker(A)$ and $c \in \mathbb{F}$. Then

$$A(cv) = c(Av) = c \cdot 0 = 0.$$

So $cv \in \ker(A)$. (ba) Assume $v_1, v_2 \in \mathrm{im}(A)$. Then there exists $w_1, w_2 \in \mathbb{F}^n$ such that $Aw_1 = v_1$ and $Aw_2 = v_2$. Then

$$A(w_1 + w_2) = Aw_1 + Aw_2 = v_1 + v_2.$$

So $v_1 + v_2 \in \mathrm{im}(A)$.
(bb) Assume $v \in \mathrm{im}(A)$ and $c \in \mathbb{F}$. Then there exists $w \in \mathbb{F}^n$ such that $Aw = v$. Let $z = cw$. Then $Az = A(cw) = c(Aw) = cv$. So $cv \in \mathrm{im}(A)$. □

3.4.3 Comparing kernel and image of A and PAQ

Proposition 3.12. *Let $\mathbb{F}$ be a field and let $A \in M_{m\times n}(\mathbb{F})$. Let $P^{-1} \in GL_m(\mathbb{F})$ and $Q^{-1} \in GL_n(\mathbb{F})$. Then*

$$\ker(P^{-1}AQ^{-1}) = Q\ker(A) \qquad \textit{and} \qquad \mathrm{im}(P^{-1}AQ^{-1}) = P^{-1}\,\mathrm{im}(A).$$

Proof. The proof of the first equality is

$$\begin{aligned}\ker(P^{-1}AQ^{-1}) &= \{v \in \mathbb{F}^n \mid P^{-1}AQ^{-1}v = 0\} \\ &= \{v \in \mathbb{F}^n \mid AQ^{-1}v = P0\} \\ &= \{QQ^{-1}v \in \mathbb{F}^n \mid AQ^{-1}v = 0\} \\ &= \{Qw \in \mathbb{F}^n \mid Aw = 0\} \\ &= Q\{w \in \mathbb{F}^n \mid Aw = 0\} = Q\ker(A),\end{aligned}$$

and the proof of the second equality is

$$\begin{aligned}\mathrm{im}(P^{-1}AQ^{-1}) &= \{P^{-1}AQ^{-1}v \mid v \in \mathbb{F}^n\} \\ &= P^{-1}\{AQ^{-1}v \mid v \in \mathbb{F}^n\} \\ &= P^{-1}\{AQ^{-1}v \mid QQ^{-1}v \in \mathbb{F}^n\} \\ &= P^{-1}\{Aw \mid Qw \in \mathbb{F}^n\} \\ &= P^{-1}\{Aw \mid w \in \mathbb{F}^n\} = P^{-1}\mathrm{im}(A).\end{aligned}$$

□

3.4.4 Comparing kernel and image of A and 1_r

Proposition 3.13. *Let $A \in M_{m\times n}(\mathbb{F})$. Let $r \in \{1, \ldots, \min(m, n)\}$ and $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ such that $A = P\mathbf{1}_r Q$. Then*

$$\ker(A) = Q^{-1}\ker(1_r) \qquad \textit{and} \qquad \mathrm{im}(A) = P\,\mathrm{im}(1_r).$$

Proof. By Proposition 3.3,

$$\ker(A) = Q^{-1}\ker(P^{-1}AQ^{-1}) = Q^{-1}\ker(1_r) \qquad \text{and} \qquad \mathrm{im}(A) = P\,\mathrm{im}(P^{-1}AQ^{-1}) = P\,\mathrm{im}(1_r).$$

□

3.4.5 Comparing $\dim(\ker(A))$ and $\dim(\mathrm{im}(A))$

Proposition 3.14. *Let $A \in M_{m\times n}(\mathbb{F})$. Then*

$$\dim(\mathrm{im}(A)) = (\textit{number of columns of } A) - \dim(\ker(A)).$$

Proof. Let $r \in \{1, \ldots, \min(m, n)\}$ and $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ such that $A = P\mathbf{1}_r Q$. By Proposition 3.4 and (kerimbasis),

$$\begin{aligned}\dim(\mathrm{im}(A)) &= \dim(P\,\mathrm{im}(1_r)) = \dim(\mathrm{im}(1_r)) = r \\ &= n - (n-r) = (\text{number of columns of } A) - \dim(\ker(1_r)) \\ &= (\text{number of columns of } A) - \dim(Q^{-1}\ker(1_r)) \\ &= (\text{number of columns of } A) - \dim(\ker(A)).\end{aligned}$$

□

3.4.6 Solutions of a system when A is invertible

Proposition 3.15. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$. If $m = n$ and $A \in GL_n(\mathbb{F})$ then*

$$\mathrm{Sol}(Ax = b) = \{A^{-1}b\}.$$

Proof. Assume A is invertible and $Ax = b$. Then $x = 1x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$. □

3.4.7 Solutions of a system from a single one and the $\ker(A)$

Proposition 3.16. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$ and assume $\mathrm{Sol}(Ax = b) \neq \emptyset$. Let $p \in \mathrm{Sol}(Ax = b)$. Then*

$$\mathrm{Sol}(Ax = b) = p + \ker(A).$$

Proof. Assume $\mathrm{Sol}(Ax = b) \neq \emptyset$ and let $p \in \mathrm{Sol}(Ax = b)$.
To show: (a) $\mathrm{Sol}(Ax = b) \subseteq p + \ker(A)$.
(b) $p + \ker(A) \subseteq \mathrm{Sol}(ax = b)$.

(a) Let $q \in Sol(Ax = b)$.
Then $q - p \in \ker(A)$.
So $q \in p + \ker(A)$.
So $Sol(Ax = b) \subseteq p + \ker(A)$.

[(b) Let $k \in \ker(A)$.
Then $A(p + k) = Ap + Ak = b + 0 = b$.
So $p + k \in p + \ker(A)$.
So $p + \ker(A) \subseteq \mathrm{Sol}(Ax = b)$.

So $p + \ker(A) = \mathrm{Sol}(Ax = b)$ □

3.4.8 All solutions of a system $Ax = b$

Proposition 3.17. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^n$. Assume $r \in \{1, \ldots, \min(m, n)\}$ and $P \in GL_n(\mathbb{F})$ and $Q \in GL_m(\mathbb{F})$ are such that*

$$A = P1_rQ.$$

(a) If there exists $j \in \{r + 1, \ldots, m\}$ such that $(P^{-1}b)_j \neq 0$ then $\mathrm{Sol}(Ax = b) = \emptyset$.

(b) If $\mathrm{Sol}(Ax = b) \neq \emptyset$ then

$$\mathrm{Sol}(Ax = b) = Q^{-1}((P^{-1}b)_1, \ldots, (P^{-1}b)_r, 0, \ldots, 0)^t + \mathrm{span}\{q_{r+1}, \ldots, q_n\},$$

where $q_1, \ldots, q_n$ are the columns of Q^{-1}.

Proof.

$$\begin{aligned}
\mathrm{Sol}(Ax = b) = \{x \in \mathbb{F}^n \mid Ax = b\} &= \{x \in \mathbb{F}^n \mid P1_rQx = b\} \\
&= Q^{-1}\{Qx \in \mathbb{F}^n \mid P1_rQx = b\} \\
&= Q^{-1}\{y \in \mathbb{F}^n \mid P1_ry = b\} \\
&= Q^{-1}\{y \in \mathbb{F}^n \mid 1_ry = P^{-1}b\} \\
&= Q^{-1}\mathrm{Sol}(1_ry = P^{-1}b).
\end{aligned}$$

Let $z = P^{-1}b$. There are two cases:
Case 1. If there exists $j \in \{r+1, \ldots, n\}$ such that $z_j \neq 0$ then $\mathrm{Sol}(1_r y = z) = \emptyset$.
Case 2. If $z_{r+1} = \cdots = z_m = 0$ then

$$\mathrm{Sol}(1_r y = z) = (z_1, \ldots, z_r, 0, \ldots, 0)^t + \mathrm{span}\{e_{r+1}, \ldots e_n\}$$

and

$$\begin{aligned} Q^{-1}\mathrm{Sol}(1_r y = P^{-1}b) &= Q^{-1}\mathrm{Sol}(1_r y = z) \\ &= Q^{-1}(z_1, \ldots, z_r, 0, \ldots, 0)^t + Q^{-1}\mathrm{span}\{e_{r+1}, \ldots e_n\} \\ &= Q^{-1}((P^{-1}b)_1, \ldots, (P^{-1}b)_r, 0, \ldots, 0)^t + \mathrm{span}\{q_{r+1}, \ldots, q_n\}. \end{aligned}$$

□