1.2 Some proofs

Proposition 1.5. *Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.*

(a) If $A, B, C \in M_{m\times n}(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_{m\times n}(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_{m\times n}(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_{m\times n}(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A \in M_{m\times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$ then $c_1 \cdot (c_2 \cdot A) = (c_1c_2) \cdot A$.

(f) If $A \in M_{m\times n}(\mathbb{F})$ and $1 \in \mathbb{F}$ is the identity in $\mathbb{F}$ then $1 \cdot A = A$.

Proof.

(a) To show: $(A + B) + C = A + (B + C)$.
To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((A + B) + C)(i, j) = (A + (B + C))(i, j)$.
Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
To show: $((A + B) + C)(i, j) = (A + (B + C))(i, j)$.

$$\begin{aligned}((A + B) + C)(i, j) &= (A + B)(i, j) + C(i, j) = (A(i, j) + B(i, j)) + C(i, j)\\ &= A(i, j) + (B(i, j) + C(i, j)), \qquad \text{since addition is associative in } \mathbb{F},\\ &= A(i, j) + (B + C)(i, j) = (A + (B + C))(i, j).\end{aligned}$$

(b) To show: $A + B = B + A$.
To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + B)(i, j) = (B + A)(i, j)$.
Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
To show: $(A + B)(i, j) = (B + A)(i, j)$.

$$\begin{aligned}(A + B)(i, j) &= A(i, j) + B(i, j)\\ &= B(i, j) + A(i, j), \qquad \text{since } \mathbb{F} \text{ has commutative addition},\\ &= (B + A)(i, j).\end{aligned}$$

(c) To show: (ca) $0 + A = A$.
(cb) $A + 0 = A$.

(ca) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(0 + A)(i, j) = A(i, j)$.
Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
To show: $(0 + A)(i, j) = A(i, j)$.

$$(0 + A)(i, j) = 0(i, j) + A(i, j) = 0 + A(i, j) = A(i, j).$$

(cb) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + 0)(i, j) = A(i, j)$.
Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
To show: $(A + 0)(i, j) = A(i, j)$.

$$(A + 0)(i, j) = A(i, j) + 0(i, j) = A(i, j) + 0 = A(i, j).$$

(d) To show: (da) $A + (-A) = 0$.

(db) $(-A) + A = 0$.

(da) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(A + (-A))(i, j) = 0(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $(A + (-A))(i, j) = 0(i, j)$.

$$\begin{aligned}(A + (-A))(i, j) &= A(i, j) + (-A)(i, j) = A(i, j) + (-A(i, j)) \\ &= 0 = 0(i, j).\end{aligned}$$

(db) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $((-A) + A)(i, j) = 0(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $((-A) + A)(i, j) = 0(i, j)$.

$$\begin{aligned}((-A + A)(i, j) &= (-A)(i, j) + A(i, j) = (-A(i, j)) + A(i, j) \\ &= 0 = 0(i, j).\end{aligned}$$

(e) Assume $A \in M_{m\times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$.
To show $c_1 \cdot (c_2 \cdot A) = (c_1c_2) \cdot A$.
To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(c_1 \cdot (c_2 \cdot A))(i, j) = ((c_1c_2) \cdot A)(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $(c_1 \cdot (c_2 \cdot A))(i, j) = ((c_1c_2) \cdot A)(i, j)$.

$$\begin{aligned}(c_1 \cdot (c_2 \cdot A))(i, j) &= c_1 \cdot ((c_2 \cdot A)(i, j)) = c_1 \cdot (c_2 \cdot A(i, j)) \\ &= (c_1 \cdot c_2) \cdot A(i, j)) = ((c_1 \cdot c_2) \cdot A)(i, j)\end{aligned}$$

(f) Assume $A \in M_{m\times n}(\mathbb{F})$ and $1 \in \mathbb{F}$.
To show: (fa) $1 \cdot A = A$,
(fb) $A \cdot 1 = A$.

(fa) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(1 \cdot A)(i, j) = A(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $(1 \cdot A)(i, j) = A(i, j)$.

$$(1 \cdot A)(i, j) = 1 \cdots A(I, j) = A(i, j), \quad \text{since 1 is the identity in } \mathbb{F}.$$

(fb) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(A \cdot 1)(i, j) = A(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $(A \cdot 1)(i, j) = A(i, j)$.

$$(A \cdot 1)(i, j) = A(i, j) \cdot 1 = A(i, j), \quad \text{since 1 is the identity in } \mathbb{F}.$$

□

Proposition 1.6. *Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B, C \in M_n(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_n(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_n(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_n(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A, B, C \in M_n(\mathbb{F})$ then $A(BC) = (AB)C$.

(f) If $A, B, C \in M_n(\mathbb{F})$ then $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.

(g) If $A \in M_n(\mathbb{F})$ then $1A = A$ and $A1 = A$.

Proof. Parts (a), (b) (c) and (d) are the cases $m = n$ of Proposition 1.1.

(e) To show: $(AB)C = A(BC)$.
To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $((AB)C)(i, j) = (A(BC))(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $((AB)C)(i, j) = (A(BC))(i, j)$.

$$\begin{aligned}
((AB)C)(i, j) &= \sum_{k=1}^{n} (AB)(i, k)C(k, j) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} (A(i, l)B(l, k))C(k, j) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} A(i, l)(B(l, k)C(k, j)) \\
&= \sum_{l=1}^{n} A(i, l)(BC)(l, j) \\
&= (A(BC))(i, j).
\end{aligned}$$

(f) To show: (fa) $(A + B)C = AC + BC$.
(fb) $C(A + B) = CA + CB$.

(fa) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $((A + B)C)(i, j) = (AC + BC)(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $((A + B)C)(i, j) = (AC + BC)(i, j)$.

$$\begin{aligned}
((A + B)C)(i, j) &= \sum_{k=1}^{n} (A + B)(i, k)C(k, j) = \sum_{k=1}^{n} (A(i, k) + B(i, k))C(k, j) \\
&= \sum_{k=1}^{n} A(i, k)C(k, j) + B(i, k)C(k, j), \\
&= \sum_{k=1}^{n} A(i, k)C(k, j) + \sum_{k=1}^{n} B(i, k)C(k, j), \\
&= (AC)(i, j) + (BC)(i, j) = ((AC) + (BC))(i, j) \\
&= (AC + BC)(i, j).
\end{aligned}$$

where the third equality follows from the distributive property in $\mathbb{F}$.

(fb) To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(C(A + B))(i, j) = (CA + CB)(i, j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

To show: $(C(A+B))(i,j) = (CA+CB)(i,j)$.

$$\begin{aligned}
(C(A+B))(i,j) &= \sum_{k=1}^{n} C(i,k)(A+B)(k,j) = \sum_{k=1}^{n} C(i,k)(A(k,j)+B(k,j)) \\
&= \sum_{k=1}^{n} C(i,k)A(k,j) + C(i,k)B(k,j), \\
&= \sum_{k=1}^{n} C(i,k)A(k,j) + \sum_{k=1}^{n} C(i,k)A(k,j), \\
&= (CA)(i,j) + (CB)(i,j) = ((CA)+(CB))(i,j) \\
&= (CA+CB)(i,j).
\end{aligned}$$

where the third equality follows from the distributive property in $\mathbb{F}$.

(g) To show: (ga) $1 \cdot A = A$,
(gb) $A \cdot 1 = A$.

(ga) To show: If $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$ then $(1 \cdot A)(i,j) = A(i,j)$.
Assume $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$.
To show: $(1 \cdot A)(i,j) = A(i,j)$.

$$\begin{aligned}
(1 \cdot A)(i,j) &= \sum_{k=1}^{n} 1(i,k)A(k,j) = 1(i,i)A(i,j) + \sum_{\substack{k\in\{1,\ldots,n\}\\ k\neq i}} 1(i,k)A(k,j) \\
&= 1 \cdot A(i,j) + 0 = A(i,j).
\end{aligned}$$

(gb) To show: If $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$ then $(A \cdot 1)(i,j) = A(i,j)$.
Assume $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$.
To show: $(A \cdot 1)(i,j) = A(i,j)$.

$$\begin{aligned}
(A \cdot 1)(i,j) &= \sum_{k=1}^{n} A(i,k)1(k,j) = A(i,j)1(j,j) + \sum_{\substack{k\in\{1,\ldots,n\}\\ k\neq j}} A(i,k)1(k,j) \\
&= 0 + A(i,j) \cdot 1 = A(i,j).
\end{aligned}$$

□

Proposition 1.7. *Let $m,n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$, and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B \in M_{m\times n}(\mathbb{F})$ then $(A+B)^t = A^t + B^t$,

(b) If $A \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$ then $(c \cdot A)^t = c \cdot A^t$,

(c) If $A, B \in M_n(\mathbb{F})$ then $(AB)^t = B^t A^t$.

(d) If $A \in M_n(\mathbb{F})$ then $(A^t)^t = A$.

Proof.

(a) Assume $A \in M_{m\times n}(\mathbb{F})$.
To show $(A+B)^t = A^t + B^t$.

To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $(A+B)^t(i,j) = (A^t + B^t)(i,j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $(A+B)^t(i,j) = (A^t + B^t)(i,j)$.

$$\begin{aligned}(A+B)^t(i,j) &= (A+B)(j,i) = A(j,i) + B(j,i) \\ &= A^t(i,j) + B^t(i,j) = (A+B)^t(i,j).\end{aligned}$$

(b) Assume $A \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$.
To show $(c \cdot A)^t = c \cdot A^t$.
To show: If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ then $((c \cdot A)^t)(i,j) = (c \cdot A^t)(i,j)$.
Assume $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
To show: $((c \cdot A)^t)(i,j) = (c \cdot A^t)(i,j)$.

$$((c \cdot A)^t)(i,j) = (c \cdot A)(j,i) = c \cdot A(j,i) = c \cdot A^t(i,j) = (c \cdot A^t)(i,j)$$

(c) Assume $A, B \in M_n(\mathbb{F})$.
To show: $(AB)^t = B^t A^t$.
To show: If $i, j \in \{1, \ldots, n\}$ then $(AB)^t(i,j) = (B^t A^t)(i,j)$.
Assume $i, j \in \{1, \ldots, n\}$.
To show: $(AB)^t(i,j) = (B^t A^t)(i,j)$.

$$\begin{aligned}(AB)^t(i,j) = (AB)(j,i) &= \sum_{k=1}^{n} A(j,k)B(k,i) \\ &= \sum_{k=1}^{n} A^t(k,j)B^t(i,k) = \sum_{k=1}^{n} B^t(i,k)A^t(k,j) = (B^t A^t)(i,j).\end{aligned}$$

(d) Assume $A \in M_n(\mathbb{F})$.
To show: $(A^t)^t = A$.
To show: If $i, j \in \{1, \ldots, n\}$ then $((A^t)^t)(i,j) = A(i,j)$.
Assume $i, j \in \{1, \ldots, n\}$.
To show: $((A^t)^t)(i,j) = A(i,j)$.

$$((A^t)^t)(i,j) = (A^t)(j,i) = A(i,j)$$

□