2.2 Normal form for an $m \times n$ matrix

2.2.1 Normal form algorithm and the matrix 1_r

Let $\mathbb{F}$ be a field and let $A \in M_{m\times n}(\mathbb{F})$.

Step 1. Increasing the number of lower left 0 entries Order the entries in the lower triangular part of the matrix from bottom left corner, working up the first column, then taking the bottom entry of the second column and working up the second column and so forth. If the first nonzero entry in this process is b, with an entry a immediately above it then left multiply by $y_i(\frac{a}{b})^{-1}$, where i is the row number of the row that a is in.

In this way use left multiplications to successively increase the quantity of lower left 0 entries. After this process the resulting matrix will satisfy the conditions

(a) There exists $r \in \{1, \ldots, m\}$ such that rows $1, \ldots, r$ contain a nonzero entry and rows $r+1, \ldots, m$ are all 0,

(d) If $i \in \{1, \ldots, r-1\}$ and (i, c_i) is the position of the first nonzero entry in row i then $c_1 < c_2 < \cdots < c_r$.

Step 2. Making the first nonzero entry in each row equal to 1. Left multiply by $h(d_1, \ldots, d_m)^{-1}$ where d_i is the first nonzero entry in row i and 1 otherwise. This makes the first nonzero entry in each nonzero row equal to 1.

Step 3. Making the entries above first nonzero entry in each row equal to 0. Then order the entries that sit above first nonzero entries in a row, down columns, working left to right. If an entry in position (i, j) is equal to $a_{ij} \neq 0$ then left multiply by $x_{ij}(a_{ij})^{-1}$. This will zero out this entry. Successively doing this reduces the matrix to a matrix which satisfes the properties

(a) There exists $r \in \{1, \ldots, m\}$ such that rows $1, \ldots, r$ contain a nonzero entry and rows $r+1, \ldots, m$ are all 0,

(b) If $i \in \{1, \ldots, r-1\}$ then the first nonzero entry in each row i is 1,

(c) If $i \in \{1, \ldots, r-1\}$ and (i, c_i) is the position of the first nonzero entry in row i then all other entries in column c_i are 0,

(d) If $i \in \{1, \ldots, r-1\}$ and (i, c_i) is the position of the first nonzero entry in row i then $c_1 < c_2 < \cdots < c_r$.

Step 4. Making the remaining upper right entries all equal to 0. Then order upper triangular entries left to right in each row, and work on rows top to bottom. If the entry in position (i, j) is a_{ij} and $a_{ij} \neq 0$ then right multiply by $x_{ij}(a_{ij})^{-1}$. This will make the (i, j) entry equal to 0.

Step 5. Rearranging columns to get 1_r. Right multiplying by a permutation u_J will rearrange the columns so that the first r columns consists of 1 down the diagonal. The permutation u_J is minimal length permutation sending $(1, 2, \ldots, r)$ to $(c_1, \ldots, c_r)$, where c_i is the column number of the first nonzero entry in row i. Explicitly,

$$u_J = (s_{c_1-1} \cdots s_2 s_1)(s_{c_2-1} \cdots s_3 s_2) \cdots (s_{c_r-1} \cdots s_{r+1} s_r).$$

Letting E_{ij} denote the matrix with 1 in entry (i, j) and 0 elsewhere, the resulting matrix after this process is

$$1_r \in M_{m\times n}(\mathbb{F}) \qquad \text{given by} \qquad 1_r = E_{11} + \cdots + E_{rr}. \tag{1rdefn}$$