Topic 1. Examples 2 and 3 and 4. Let us solve the equation

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Let

$$y_1(c)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \qquad \text{and} \qquad d(c_1, c_2)^{-1} = \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix}.$$

Multiply both sides of the equation by $y_1(2)^{-1}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Multiply both sides by $d(1,3)^{-1}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{which gives} \qquad \begin{matrix} y = 1, \\ x = 0 - y = 0 - 1 = -1. \end{matrix}$$

So

$$\mathrm{Sol}\left(\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

Let

$$y_1(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad d(c_1, c_2) = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = y_1(2) \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = y_1(2) d(1,3) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = y_1(2) d(1,3) x_{12}(1) \cdot 1_2.$$

Thus, if

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{then} \qquad A = P \cdot 1_2, \qquad \text{where} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $P = y_1(2) d(1,3) x_{12}(1)$ in $M_{2\times 2}(\mathbb{Q})$.

Remark 10.1. The process of solving the equation has computed

$$A^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = P^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = d(1,3)^{-1} y_1(2)^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

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