Topic 2. Example 10. If $Ax = b$ is

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

then multiplying on the left by $(y_2(0)y_1(1)y_2(2))^{-1} = y_2(2)^{-1}y_1(1)^{-1}y_2(0)^{-1}$ gives

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \qquad \text{giving} \qquad \begin{matrix} x_1 = 5 + 3x_2, \\ x_2 = -2 - x_3, \end{matrix}$$

so that $x_1 = 5 + 3(-2 - x_3) = -1 - 3x_3$,

$$\begin{matrix} x_1 = -3 - 3x_3, \\ x_2 = -2 - x_3, \\ x_3 = 0 + x_3, \end{matrix} \qquad \text{and} \qquad \mathrm{Sol}(Ax = b) = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Topic 2. Example 11. If

$$A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix} = h(2, 3, -1) \begin{pmatrix} 1 & 3 & \frac{9}{2} \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 1 \end{pmatrix} = h(2, 3, -1)x_{12}(3)x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3}) \cdot 1_3,$$

giving $\mathrm{rank}(A) = 3$ and

$$\det(A) = \det(h(2, 3, -1)) \det(x_{12}(3)) \det\left(x_{13}(\tfrac{9}{2})\right) \det\left(x_{23}(\tfrac{8}{3})\right) = (2 \cdot 3 \cdot (-1)) \cdot 1 \cdot 1 \cdot 1 = -6.$$

Topic 2. Example 12. Since

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \\ &= y_1(-1) \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= y_1(-1)y_2(3) \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix} \\ &= y_1(-1)y_2(3)h(-1, 2, -7) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \\ &= y_1(1)y_2(3)h(-1, 2, -7)x_{12}(-1)x_{13}(-1)x_{23}(\tfrac{3}{2}) \cdot 1_3 \end{aligned}$$

then $\mathrm{rank}(A) = 3$ and

$$\det(A) = (-1) \cdot (-1) \cdot ((-1) \cdot 2 \cdot (-7)) \cdot 1 \cdot 1 \cdot 1 = 7.$$

Topic 2. Example 13. Since $1 = AA^{-1}$ then

$$\det(1) = \det(A)\det(A^{-1}) \quad \text{and} \quad 1 = \det(A)\det(A^{-1}) \quad \text{and} \quad \frac{1}{\det(A)} = \det(A^{-1}).$$