8 Vector geometry

8.1 $\mathbb{R}^2$

A favourite example is $\mathbb{R}^2 = \{|x_1, x_2\rangle \mid x_1, x_2 \in \mathbb{R}\}$ with *addition* and *scalar multiplication* given by

$$|x_1, x_2\rangle + |y_1, y_2\rangle = |x_1 + y_1, x_2 + y_2\rangle \qquad \text{and} \qquad c\,|x_1, x_2\rangle = |cx_1, cx_2\rangle, \quad \text{for } c \in \mathbb{R},$$

with *standard inner product*

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ (x, y) & \longmapsto & \langle x|y\rangle \end{array} \qquad \text{given by} \qquad \langle x_1, x_2 | y_1, y_2\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + x_2y_2,$$

with *length function*

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{array} \qquad \text{given by} \qquad \|\,|x_1, x_2\rangle\,\| = \sqrt{\langle x_1, x_2 | x_1, x_2\rangle} = \sqrt{x_1^2 + x_2^2},$$

and *distance function* $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$d(|x_1, x_2\rangle, |y_1, y_2\rangle) = \|\,|x_1, x_2\rangle - |y_1, y_2\rangle\| = \|\,|x_1 - y_1, x_2 - y_2\rangle\,\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

PICTURE

8.2 The vector space $\mathbb{R}^n$

Let $n \in \mathbb{Z}_{\geq 0}$. The space $\mathbb{R}^n$ is

$$\mathbb{R}^n = \{|x_1, x_2, \ldots, x_n\rangle \mid x_i \in \mathbb{R}\},$$

with *addition and scalar multiplication* given by

$$\begin{gathered} |x_1, x_2, \ldots, x_n\rangle + |y_1, y_2, \ldots, y_n\rangle = |x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle \quad \text{and} \\ c\,|x_1, x_2, \ldots, x_n\rangle = |cx_1, cx_2, \ldots, cx_n\rangle, \qquad \text{for } c \in \mathbb{R}, \end{gathered}$$

and with *standard inner product* $\langle\ ,\ \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\langle x_1, x_2, \ldots, x_n | y_1, y_2, \ldots, y_n\rangle = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots x_ny_n,$$

with *length function* $\|\ \|\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\|\,|x_1, x_2, \ldots, x_n\rangle\,\| = \sqrt{\langle (x_1, \ldots, x_n), (x_1, \ldots, x_n)\rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and *distance function* $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} d(\,|x_1, x_2, \ldots, x_n\rangle, |y_1, y_2, \ldots, y_n\rangle\,) &= \|\,|x_1, x_2, \ldots, x_n\rangle - |y_1, y_2, \ldots, y_n\rangle\,\| \\ &= \|\,|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\rangle\,\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}. \end{aligned}$$