Topic 3. Example 10. Find the Cartesian equation of the plane with vector form

$$|x,y,z\rangle = s\,|1,-1,0\rangle + t\,|2,0,1\rangle + |-1,1,1\rangle, \qquad \text{with } s,t\in\mathbb{R}.$$

A normal vector to this plane is

$$n = u\times v, \qquad \text{where } u = |1,-1,0\rangle \text{ and } v = |2,0,1\rangle.$$

Then $u\times v = |-1-0, -(1-0), 0-(-2)\rangle = |-1,-1,2\rangle$, and $|-1,1,1\rangle$ is a point in the plane, and

$$\langle -1,1,1\,|u\times v\rangle = \langle -1,1,1\,|-1,-1,2\rangle = 1-1+2 = 2.$$

Since the plane is $|-1,1,1\rangle + \{|x,y,z\rangle\in\mathbb{R}^3 \mid \langle x,y,z\,|-1-1,2\rangle = 0\}$ then the Cartesian equation of the plane is

$$-x-y+2z = 2.$$

Topic 3. Example 11. Find the vector equation for the plane in $\mathbb{R}^3$ containing the points $P = |1,0,2\rangle$ and $Q = |1,2,3\rangle$ and $R = |4,5,6\rangle$.

The point $|1,0,2\rangle$ is in the plane and two vectors in the plane are

$$Q-P = |0,2,1\rangle \qquad \text{and} \qquad R-P = |3,5,4\rangle.$$

So the points in the plane are the points $|x,y,z\rangle$ in $\mathbb{R}^3$ which satisfy

$$|x,y,z\rangle = |1,0,2\rangle + s|0,2,1\rangle + t|3,5,4\rangle \qquad \text{with } s,t\in\mathbb{R}.$$

Topic 3. Example 12. Where does the line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

intersect the plane $3x+2y+z=20$?

The line in parametric form is

$$\begin{aligned} x &= 1+t, \\ y &= 2+2t, \qquad \text{with } t\in\mathbb{R}, \\ z &= 3+3t, \end{aligned}$$

and plugging into the equation of the plane gives

$$20 = 3(t+1)+2(2t+2)+(3t+3) = 10t+10 \qquad \text{so that} \qquad t=1.$$

Thus the point $|x,y,z\rangle$ with $x=1+1=2$, $y=2+2=4$ and $z=3+3$ is on both the line and the plane.