Topic 3. Example 5. Find a vector perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$.
By definition of the cross product

$$|1,1,1\rangle \times |1,-1,-2\rangle = |\,1\cdot(-2), -1\cdot(-1), -(1\cdot(-2)-1\cdot 1), 1\cdot(-1)-1\cdot 1\,\rangle = |-1,3,-2\rangle.$$

The vector $|-1,3,-2\rangle$ is perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$ since

$$\langle -1,3,-2\,|\,1,1,1\rangle = -1+3-2=0 \qquad \text{and} \qquad \langle -1,3,-2\,|\,1,-1,-2\rangle = -1-3+4=0.$$

An even better way to answer this question is to find *all* vectors $|a,b,c\rangle$ that are perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$. These are the vectors $|a,b,c\rangle$ such that

$$\begin{array}{l}\langle a,b,c|1,1,1\rangle = 0,\\ \langle a,b,c|1,-1,-2\rangle = 0,\end{array} \qquad \text{so that} \qquad \begin{array}{l}a+b+c=0,\\ a-b-2c=0.\end{array}$$

In matrix form these equations are

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & -2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Multiplying both sides by $\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$ gives

$$\begin{pmatrix}1 & -1 & -2\\ 0 & 2 & 3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Multiplying both sides by $\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ gives

$$\begin{pmatrix}1 & -1 & -2\\ 0 & 1 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Multiplying both sides by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ gives

$$\begin{pmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

So

$$\begin{array}{l}a-\frac{1}{2}c=0,\\ b+\frac{3}{2}c=0,\end{array} \qquad \text{which gives} \qquad \begin{array}{l}a=\frac{1}{2}c,\\ b=-\frac{3}{2}c,\\ c=c\end{array}$$

So the vectors $|a,b,c\rangle$ that are perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$ are the vectors in

$$\text{span}\left\{\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\\ 1\end{pmatrix}\right\} = \{t\cdot|\tfrac{1}{2},-\tfrac{3}{2},1\rangle \mid t\in\mathbb{R}\}.$$