Topic 4. Example 13. In $\mathbb{R}^3$, is $|1,2,3\rangle \in \mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$?

To show: There exist $c_1, c_2 \in \mathbb{R}$ such that $c_1|1,-1,2\rangle + c_2|-1,1,2\rangle = |1,2,3\rangle$.
To show: The system

$$\begin{aligned} c_1 - c_2 &= 1, \\ -c_1 + c_2 &= 2, \\ 2c_1 + 2c_2 &= 3, \end{aligned} \qquad \text{has a solution.}$$

In matrix form the equations are $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

Multiplying both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ gives $\begin{pmatrix} 2 & 2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.
Already this gives an equation $0c_1 + 0c_2 = 3$, which has no solution.
So $|1,2,3\rangle \notin \mathbb{R}\text{-span}\{|1,-1,2\rangle$ and $|-1,1,2\rangle\}$.
So $|1,2,3\rangle$ is not a linear combination of $|1,-1,2\rangle$ and $|-1,1,2\rangle$.

Topic 4. Example 14. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$.?

To show: There exist $c_1, c_2 \in \mathbb{R}$ such that $c_1(1 + x + x^2) + c_2(3 + x^2) = 1 - 2x - x^2$.
To show: The system

$$\begin{aligned} c_1 + 3c_2 &= 1, \\ c_1 + 0c_2 &= -2, \\ c_1 + c_2 &= -1. \end{aligned} \qquad \text{has a solution.}$$

In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Skipping the row reduction steps, $c_1 = -2$ and $c_2 = 1$ is a solution to this system.
So $-2(1 + x + x^2) + (3 + x^2) = 1 - 2x - x^2$.
So $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$.
So $1 - 2x - x^2$ is a linear combination of $1 + x + x^2$ and $3 + x^2$.

Topic 4. Example 15. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1), (2,2,2), (3,3,3)\}. \qquad \text{Determine span}(S).$$

In this case

$$\begin{aligned} \text{span}(S) &= \{c_1\,|1,1,1\rangle + c_2\,|2,2,2\rangle + c_3\,|3,3,3\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\ &= \{c_1\,|1,1,1\rangle + 2c_2\,|1,1,1\rangle + 3c_3\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\ &= \{(c_1 + 2c_2 + 3c_3)\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\ &= \{t\,|1,1,1\rangle \mid t \in \mathbb{R}\} \\ &= \{|t,t,t\rangle \mid t \in \mathbb{R}\}. \end{aligned}$$