Topic 4. Example 17. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{|1,2,0\rangle, |1,5,3\rangle, |0,1,1\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^3.$$

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar mutliplication then $\mathrm{span}(S) \subseteq \mathbb{R}^2$.

(b) To show: $\mathbb{R}^2 \subseteq \mathrm{span}(S)$.
To show: $\mathrm{span}\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathrm{span}(S)$.
Since $\mathrm{span}(S)$ is closed under addition and scalar multiplication,
To show: $\{|1,0.0\rangle, |0,1.0\rangle, |0,0,1\rangle\} \subseteq \mathrm{span}(S)$.
To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle &= |1,0,0\rangle, \\ d_1|1,2,0\rangle + d_2|1,5,3\rangle + d_3|0,1,1\rangle &= |0,1,0\rangle, \\ r_1|1,2,0\rangle + r_2|1,5,3\rangle + r_3|0,1,1\rangle &= |0,0,1\rangle, \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there do not exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{|1,0.0\rangle, |0,1.0\rangle, |0,0,1\rangle\} \not\subseteq \mathbb{R}\text{-span}(S)$.
So $\mathrm{span}(S) \neq \mathbb{R}^2$.