Topic 4. Example 23. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be given by

$$v_1 = |1,2,3\rangle, \quad v_2 = |3,6,9\rangle, \quad v_3 = |-1,0,-2\rangle, \quad v_4 = (1,4,4).$$

(a) Is $\{v_1, v_2, v_3, v_4\}$ linearly independent?
(b) Express v_2 and v_4 as linear combinations of v_1 and v_3.
(c) Is $\{v_1, v_3\}$ linearly independent?

(a) Since $v_2 = 3v_1$ then $3v_1 - v_2 = 0$.
So $c_1 = 3$, $c_2 = -1$, $c_3 = 0$, $c_4 = 0$ is a solution to $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.
So $\{v_1, v_2, v_3, v_4\}$ is not linearly independent.

(b) Since $v_2 = 3v_1 + 0v_3$ then $v_2 \in \mathbb{R}\text{-span}\{v_1, v_3\}$.
Since $v_1 + v_3 = (0,2,1)$ and $v_1 + (0,2,1) = (1,4,4)$.
So $v_4 = 2v_1 + v_3$. So $v_4 \in \mathbb{R}\text{-span}\{v_1, v_3\}$.

(c) To show: If $c_1, c_2 \in \mathbb{R}$ and $c_1|1,2,3\rangle + c_2|-1,0,2\rangle = |0,0,0\rangle$ then $c_1 = 0$ and $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{R}$ and $c_1|1,2,3\rangle + c_2|-1,0,2\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} c_1 - c_2 &= 0, \\ 2c_1 + 0c_2 &= 0, \\ 3c_1 + 2c_2 &= 0, \end{aligned} \qquad \text{or, equivalently,} \qquad \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(Ex23c)}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has only one solution: $c_1 = 0$, $c_2 = 0$.
So $\{v_1, v_3\}$ is linearly independent.