Topic 4. Example 9. Is $W = \{a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}[x]_{\le 2}$?

A *subspace of* $\mathbb{R}[x]_{\le 2}$ is a subset $W \subseteq \mathbb{R}[x]_{\le 2}$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof.

(a) Assume $w_1 = a_1x + a_2x^2 \in W$ and $w_2 = b_1x + b_2x^2 \in W$.
Then $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.
Then $w_1+w_2 = a_1x+a_2x^2+b_1x+b_2x^2 = (a_1+a_1)x+(b_1+b_2)x^2$ and $a_1+b_1 \in \mathbb{R}$ and $a_2+b_2 \in \mathbb{R}$.
So $w_1 + w_2 \in W$.
(b) $0 = 0x + 0x^2$ satisfies $0 \in \mathbb{R}$ and $0 \in \mathbb{R}$. So $0 \in W$.
(c) Assume $w = a_1x + a_2x^2 \in W$.
Then $a_1, a_2 \in \mathbb{R}$.
Then $-w = -(a_1x + a_2x^2) = -a_1x + (-a_2)x^2$ and $-a_1 \in RR$ and $-a_2 \in \mathbb{R}$.
So $-w \in W$.
(d) Assume $w = a_1x + a_2x^2 \in W$ and $c \in \mathbb{R}$.
Then $a_1, a_2 \in \mathbb{R}$.
Then $cw = c(a_1x + a_2x^2) = (ca_1)x + (ca_2)x^2$ and $ca_1 \in \mathbb{R}$ and $ca_2 \in \mathbb{R}$.
So $cw \in W$.

So W is a subspace of $\mathbb{R}[x]_{\le 2}$. □

Topic 4. Example 11. Is

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; ad - bc = 0 \right\} \qquad \text{a subspace of } M_2(\mathbb{R})?.$$

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $S \subseteq M_{2\times 2}(\mathbb{R})$ such that

(a) If $w_1, w_2 \in S$ then $w_1 + w_2 \in S$,
(b) $0 \in S$,
(c) If $w \in S$ then $-w \in S$,
(d) If $w \in S$ and $c \in \mathbb{R}$ then $cw \in S$.

Let $w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $1 \cdot 0 - 0 \cdot 0 = 0 - 0 = 0$ then $w_1 \in S$.
Let $w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $0 \cdot 1 - 0 \cdot 0 = 0 - 0 = 0$ then $w_2 \in S$.
Then

$$w_1 + w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad 1 \cdot 1 - 0 \cdot 0 = 1.$$

So $w_1 + w_2 \notin S$.
So S is not a subspace of $M_{2\times 2}(\mathbb{R})$.