10.3 Gram matrix of $\langle,\rangle$ with respect to a basis B

Assume $n \in \mathbb{Z}_{>0}$ and $\dim(V) = n$. Let $\langle,\rangle\colon V \times V \to \mathbb{F}$ be a bilinear form and let $B = \{b_1, \dots, b_n\}$ be a basis of V. The *Gram matrix of* $\langle,\rangle$ *with respect to the basis* B is

$$G_B \in M_n(\mathbb{F}) \qquad \text{given by} \qquad G_B(i,j) = \langle b_i, b_j\rangle.$$

Let $C = \{c_1, \dots, c_n\}$ be another basis of V and let P_{CB} be the change of basis matrix given by

$$c_i = \sum_{i=1}^{n} P_{BC}(j,i) b_j, \qquad \text{for } i \in \{1, \dots, n\}.$$

Since

$$G_C(i,j) = \langle c_i, c_j\rangle = \sum_{k,l=1}^{n} \langle P_{BC}(k,i)b_k, P_{BC}(l,j)b_l\rangle = \sum_{k,l=1}^{n} P_{BC}(k,i)G_B(k,l)P_{BC}(l,j),$$

then

$$G_C = P_{BC}^t G_B P_{BC},$$

10.4 Quadratic forms

Let $\mathbb{F}$ be a field, V an $\mathbb{F}$-vector space and $\langle,\rangle\colon V \times V \to \mathbb{F}$ a bilinear form. The *quadratic form associated to* $\langle,\rangle$ is the function

$$\|\ \|^2\colon V \to \mathbb{F} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

Theorem 10.1. *Let V be a vector space over a field $\mathbb{F}$ and let $\langle,\rangle\colon V \times V \to \mathbb{F}$ be a bilinear form. Let $\|\ \|^2\colon V \to \mathbb{F}$ be the quadratic form associated to $\langle,\rangle$.*

(a) (Parallelogram property) If $x, y \in V$ then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) (Pythagorean theorem) If $x, y \in V$ and $\langle x,y\rangle = 0$ and $\langle y,x\rangle = 0$ then

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2.$$

(c) (Reconstruction) Assume that $\langle,\rangle$ is symmetric and that $2 \neq 0$ in $\mathbb{F}$. Let $x, y \in V$. Then

$$\langle x,y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2).$$

Theorem 10.2. *Let $\mathbb{F}$ be a field with an involution $\overline{}\colon \mathbb{F} \to \mathbb{F}$ such that the fixed field*

$$\mathbb{K} = \{a \in \mathbb{F} \mid a = \bar{a}\} \qquad \textit{is an ordered field.}$$

For $a \in \mathbb{K}$ define

$$|a|^2 = a\bar{a}.$$

Let V be an $\mathbb{K}$-vector space with a sesquilinear form $\langle,\rangle\colon V \times V \to \mathbb{F}$ such that

(a) If $x, y \in V$ then $\langle y,x\rangle = \overline{\langle x,y\rangle}$.

(b) If $x \in V$ then $\langle x,x\rangle \in \mathbb{K}_{\geq 0}$.

Let $\|\ \|\colon V \to \mathbb{F}$ be the corresponding quadratic form and assume that if $a \in \mathbb{K}_{\geq 0}$ then there exists a unique $c \in \mathbb{K}_{\geq 0}$ such that $c^2 = a$. Then

(c) (Cauchy-Schwarz) If $x, y \in V$ then $|\langle x,y\rangle| \leq \|x\| \cdot \|y\|$.

(d) (Triangle inequality) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$.