10.7 Orthogonal projections

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a sesquilinear form.

Let $k\in\mathbb{Z}_{>0}$ and let W be a subspace of V such that $\dim(W)=k$ and $W\cap W^\perp=0$.

Let $(w_1,\ldots,w_k)$ be a basis of W and let $(w^1,\ldots,w^k)$ be the dual basis of W (which exists by Proposition 16.3). The *orthogonal projection onto* W is the function

$$P_W\colon V\to V \qquad \text{given by} \qquad P_W(v)=\sum_{i=1}^{k}\langle v,w_i\rangle w^i.$$

The following proposition shows that P_W does not depend on which choice of basis of W is used to construct P_W.

Proposition 10.6. *(Characterization of orthogonal projection) Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a sesquilinear form. Let $k\in\mathbb{Z}_{>0}$ and let W be a subspace of V such that $\dim(W)=k$ and $W\cap W^\perp=0$. The orthogonal projection onto W is the unique linear transformation $P\colon V\to V$ such that*

(1) If $v\in V$ then $P(v)\in W$.

(2) If $v\in V$ and $w\in W$ then $\langle v,w\rangle=\langle P(v),w\rangle$.

10.8 Orthogonal projections produce orthogonal decompositions

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let $\langle,\rangle\colon V\times V\to\mathbb{F}$ be a sesquilinear form.

Let $k\in\mathbb{Z}_{>0}$ and let W be a subspace of V such that $\dim(W)=k$ and $W\cap W^\perp=0$.

The following proposition explains how the orthogonal projection onto W produces the decomposition $V=W\oplus W^\perp$.

Theorem 10.7. *Let $n\in\mathbb{Z}_{>0}$ and let V be an inner product space with $\dim(V)=n$. Let W be a subspace of V such that $W\cap W^\perp=0$. Let P_W be the orthogonal projection onto W and let $P_{W^\perp}=1-P_W$. Then*

$$P_W^2=P_W,\quad P_{W^\perp}^2=P_{W^\perp},\quad P_WP_{W^\perp}=P_{W^\perp}P_W=0,\quad 1=P_W+P_{W^\perp},$$

$$\ker(P_W)=W^\perp,\qquad \mathrm{im}(P_W)=W\quad \textit{and}\qquad V=W\oplus W^\perp.$$