Linear Algebra

What we covered last time

(1) Connection between row reduction and elementary matrices.
(2) Solving linear systems with elementary matrices.
(3) Writing the inverse of a matrix as a product of elementary matrices.
(4) Writing a matrix as a product of elementary matrices.

I owe you Topic 1 Example 10 and 11.

Two (and a bit) remarks on inverses

Inverses are for square matrices

(1) If $A \in M_{m \times m}(\mathbb{R})$ and $B \in M_{m \times m}(\mathbb{R})$ then

$$(AB)^{-1} = B^{-1}A^{-1}$$

because

$$B^{-1}A^{-1} \cdot AB = B^{-1} \cdot 1 \cdot B = B^{-1}B = 1,$$

and

$$AB \cdot B^{-1}A^{-1} = A^{-1} \cdot 1 \cdot A = A^{-1}A = 1.$$

(2) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

because

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad-bc}{ad-bc} & 0 \\ 0 & \frac{ad-bc}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ba}{ad-bc} \\ \frac{cd-dc}{ad-bc} & \frac{-cb+ad}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Topic 1

Example 10

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix} \qquad (Eq)$$

is $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}$$

02.08.2023 ③
Linear Algebra
A. Ram

Let

$$y_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad y_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad y_3(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -c \end{pmatrix}$$

Multiply both sides of (E_4) by $y_1(1)^{-1}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 6 \\ 1 \end{pmatrix}$$

i.e. $R_1 \to R_2$
$R_2 \to R_1 - R_2$.

Multiply both sides by $y_3(0)^{-1}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 6 \\ 1 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \\ 6 \end{pmatrix}$$

i.e. $R_3 \to R_4$
$R_4 \to R_3$.

Multiply both sides by $y_2(1)^{-1}$:

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&-1&0\\0&1&1&0\\0&1&0&-1\\0&0&1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}3\\7\\1\\6\end{pmatrix}$$

So

$$\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\0&0&1&1\\0&0&1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\1\\6\\6\end{pmatrix}$$

Multiply both sides by $y_3(1)^{-1}$:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&-1\end{pmatrix}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\0&0&1&1\\0&0&1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&-1\end{pmatrix}\begin{pmatrix}3\\1\\6\\6\end{pmatrix}$$

So

$$\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}3\\1\\6\\0\end{pmatrix}$$

Multiply both sides by $x_{13}(-1)^{-1}$:

$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&-1&0\\1&1&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}3\\1\\6\\0\end{pmatrix}$$

So

$$\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}9\\1\\6\\0\end{pmatrix}$$

So

$a+b=9$
$b-d=1$
$c+d=6$
$0=0$

So

$$a = 9 - d,\quad b = 1 + d,\quad c = 6 - d,\quad d = 0 + d \quad\text{and}\quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

with d any element of $\mathbb{Q}$. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}$$

Then

$$\mathrm{Sol}(Ax = b) = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; d \in \mathbb{Q} \right\}.$$

In this example $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$ and

$$x_{13}(-1)^{-1} y_3(1)^{-1} y_2(1)^{-1} y_3(0)^{-1} y_1(1)^{-1} A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiplying both sides by

$$y_1(1) y_3(0) y_2(1) x_{13}(1)$$

gives

02.08.2023 (6)
Linear algebra
A. Ram

$$A = y_1(1)\, y_3(0)\, y_2(1)\, x_{13}(1) \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 1_3\, x_{14}(1)\, x_{24}(-1)\, x_{34}(1)$$

where

$$1_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_{14}(1) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{24}(-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $x_{34}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

So

$$A = y_1(1)\, y_3(0)\, y_2(1)\, x_{13}(1)\, 1_3\, x_{14}(1)\, x_{24}(-1)\, x_{34}(1),$$

written as a product of elementary matrices.