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Linear Algebra
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<u>Linear algebra</u>

What we covered last time

(1) Properties of inverses

(2) Writing a noninvertible matrix as a product of elementary matrices

<u>A remark on inverses</u> Since

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}\begin{pmatrix} a&b&c&d\\e&f&g&h\\i&j&k&l\\m&n&o&p \end{pmatrix} = \begin{pmatrix} a & * & * & *\\ * & * & * & *\\ * & * & * & *\\ 0&0&0&0 \end{pmatrix} \neq \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

then

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}$$ is not invertible

Let $n \in \mathbb{Z}_{>0}$ and let $E_{ij} \in M_{n\times n}(\mathbb{Q})$ be the $n\times n$ matrix with 1 in the (i,j) entry and 0 elsewhere.

<u>Row operation</u>	<u>Matrix</u>
$R_i \to R_i + cR_j$ for $i \neq j$	$e_{ij}(c) = I + cE_{ij} = \begin{pmatrix} 1 & \ddots & c \\ & & 1 \end{pmatrix}$
$R_i \leftrightarrow R_j$ for $i \neq j$	$s_{ij} = I + E_{ij} + E_{ji} - E_{ii} - E_{jj}$

$$= \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 \ddots 1 & & & & \\ & 0 & & 1 & \\ & & 1 \ddots 1 & & \\ & 1 & & 0 & \\ & & & & 1 \ddots 1 \end{pmatrix} \end{array}$$

$R_i \to cR_i$ with $c \neq 0$

$$d_i(c) = 1 + (c-1)E_{ii} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \text{ith}$$

Then

$$y_i(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & 1 & \\ & & 0 & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} = x_{i,i+1}(c)\, s_{i,i+1} \quad \text{and.}$$

$$y_i(c)^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & -c & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} = s_{i,i+1}\, x_{i,i+1}(-c). \quad \text{and}$$

$$d(c_1, \dots, c_n) = \begin{pmatrix} c_1 & & 0 \\ & c_2 \ddots & \\ 0 & & c_n \end{pmatrix} = d_1(c_1) \cdots d_n(c_n).$$

<u>Determinants</u> (are for square matrices)

The determinant is a function

$$\det : M_{n \times n}(\mathbb{Q}) \to \mathbb{Q}$$

such that

$$\det(AB) = \det(A)\det(B),$$

$\det(x_{ij}(c)) = 1$, $\det(s_{ij}) = -1$, $\det(d_i(c)) = c$.

Let $1 = E_{11} + \dots + E_{nn} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$1_r = E_{11} + \dots + E_{rr} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{pmatrix}$ for $r \in \{1, \dots, n\}$.

Then

$$d(y_i(c)) = -1, \quad \det(y_i(c)^{-1}) = -1$$

$\det(1) = 1$ and $\det(1_r) = 0$ if $r < n$.

<u>Topic 1 Example 10</u> $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Row reduction gave us

$$A = y_1(1)\, y_2(0)\, y_2(1)\, y_3(1)\, x_{13}(-1) \cdot 1_3 \cdot x_{14}(1)\, x_{24}(-1)\, x_{24}(1).$$

So $\det(A) = (-1)^4 \cdot 1 \cdot 0 \cdot 1^3 = 0$ and $\mathrm{rank}(A) = 3$.

(Rank(A) is the number of nonzero rows after row reducing to row echelon form)

<u>Topic 2 Example 4</u> Using row reduction

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = y_1(2)\, d(1,-3)\, x_{12}(1) \cdot 1_2.$$

So $\mathrm{rank}(A) = 2$, and $\det(A) = (-1) \cdot 1 \cdot (-3) \cdot 1 \cdot 1 = 3$

and

$$A^{-1} = x_{12}(1)^{-1}\, d(1,-3)^{-1}\, y_1(2)^{-1} = x_{12}(-1)\, d(1, -\tfrac{1}{3})\, y_1(2)^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

Topic 2 Example 5 $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

Using row reduction gives

$$A = y_1(-1)\, y_2(1)\, d(-1,1,-1)\, x_{12}(1)\, x_{13}(-4)\, x_{23}(3) \cdot 1_3.$$

So $\mathrm{rank}(A) = 3$ and $\det(A) = (-1)^2 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1^3 \cdot 1 = 1$
and

$$A^{-1} = x_{23}(-3)\, x_{13}(4)\, x_{12}(-1)\, d(-1,1,-1)\, y_2(1)^{-1}\, y_1(-1)^{-1}$$

$$= \begin{pmatrix} -4 & -5 & -3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$ after multiplying the matrices out.

Topic 2 Example 7 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Using row reduction gives

$$A = y_1(\tfrac{1}{3})\, d(3, \tfrac{2}{3})\, x_{12}(\tfrac{4}{3}) \cdot 1_2.$$

So $\mathrm{rank}(A) = 2$ and $\det(A) = (-1) \cdot 3 \cdot \frac{2}{3} \cdot 1 \cdot 1 = -2$
and

$$A^{-1} = x_{12}(\tfrac{-4}{3})\, d(\tfrac{1}{3}, \tfrac{3}{2})\, y_1(\tfrac{1}{3})^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

after multiplying the matrices out.

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<u>Topic 2 Example 9</u> $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix}$

Using row reduction gives

$$A = y_2(0)\, y_1(1)\, y_2(2) \cdot 1_2 \cdot x_{12}(-3)\, x_{13}(5)\, x_{24}(-2).$$

So $\mathrm{rank}(A) = 2$, ~~and~~ Since A is not square $\det(A)$ and $\mathrm{rank}(A)$ are not defined.

<u>Topic 2 Example 11</u> $A = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$

Using row reduction gives

$$A = d(2, 3, -1)\, x_{12}(3)\, x_{13}(\tfrac{4}{2})\, x_{23}(\tfrac{8}{3}) \cdot 1_3.$$

So $\mathrm{rank}(A) = 3$ and $\det(A) = 2 \cdot 3 \cdot (-1) = -6.$

<u>Topic 2 Example 12</u> $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Using row reduction gives

$$A = y_1(1)\, y_2(3)\, d(-1, 2, -7)\, x_{12}(-1)\, x_{13}(-1)\, x_{23}(\tfrac{3}{2}) \cdot 1_3$$

So $\mathrm{rank}(A) = 3$ and $\det(A) = (-1)^2 \cdot (-1) \cdot 2 \cdot (-7) \cdot 1^3 \cdot 1$
$= 14$

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<u>Transpose</u> Let

$$A=\begin{pmatrix}4&2&6&3\\1&2&-3&2i\end{pmatrix}\in M_{2\times 4}(\mathbb{C})$$ and

$$B=\begin{pmatrix}-1&0&1\\2&0&1\\0&0&1\\-2i&0&1\end{pmatrix}\in M_{4\times 3}(\mathbb{C}).$$

Then
$$A^t=\begin{pmatrix}4&1\\2&2\\6&-3\\3&-2i\end{pmatrix}$$ and $$B^t=\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}$$

Then

$$AB=\begin{pmatrix}4&2&6&3\\1&2&-3&2i\end{pmatrix}\begin{pmatrix}-1&0&1\\2&0&1\\0&0&1\\-2i&0&1\end{pmatrix}=\begin{pmatrix}-4+4+0-6i&0&4+2+6+3\\-1+4+0+4&0&1+2-3+2i\end{pmatrix}$$

and

$$B^tA^t=\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}\begin{pmatrix}4&1\\2&2\\6&-3\\3&-2i\end{pmatrix}=\begin{pmatrix}-4+4+0-6i&-1+4+0+4\\0&0\\4+2+6+3&1+2-3+2i\end{pmatrix}$$

$$=(AB)^t.$$