Determinants and volumes

09.08.2013
Linear Algebra ①
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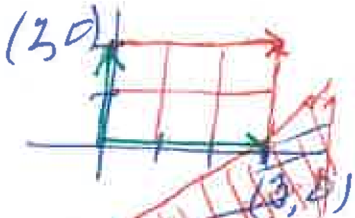
Case n=1: $|\det(2)| = 2$

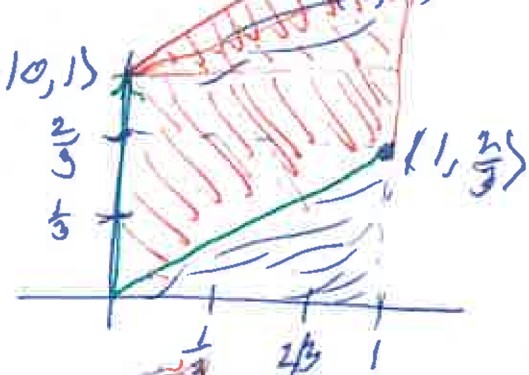
$|\det(-2)| = 2$

Case n=2:

$\left|\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right| = 3\cdot 2$



$\left|\det\begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}\right| = 1$

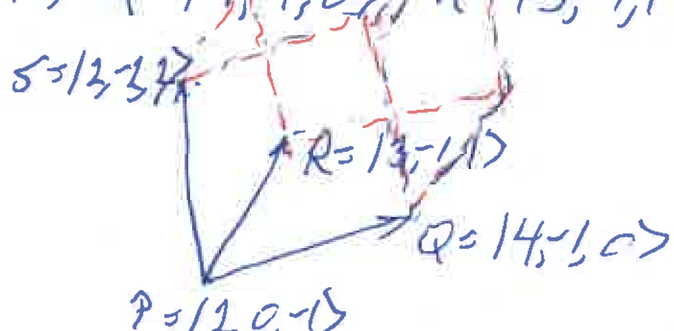


Case n=3: $P=(2,0,-1)$, $Q=(4,-1,0)$, $R=(3,-1,1)$, $S=(2,-2,2)$

$\overrightarrow{PQ} = Q-P = (2,1,1)$

$\overrightarrow{PR} = R-P = (1,-1,2)$

$\overrightarrow{PS} = S-P = (0,-2,3)$



If B is the parallelipiped with edges $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}$

then $\left|\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix}\right| = 3 = \mathrm{Vol}(B)$

In n-dimensions,

$$\mathrm{Vol}\begin{pmatrix} \text{parallelipiped in } \mathbb{R}^n \\ \text{with edge vectors} \\ e_1, \ldots, e_n \end{pmatrix} = \left|\det\begin{pmatrix} - & e_1 & - \\ - & e_2 & - \\ & \vdots & \\ - & e_n & - \end{pmatrix}\right|$$

Cross products

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Linear Algebra
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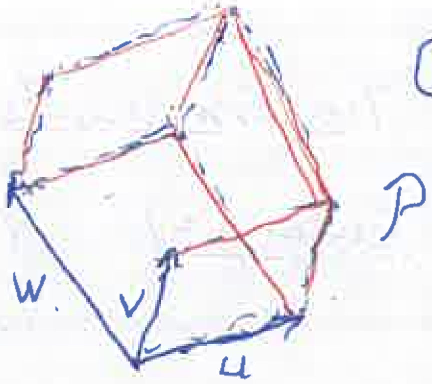


Let $u = \langle u_1, u_2, u_3 \rangle$
$v = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$.
$w = \langle w_1, w_2, w_3 \rangle$

The cross product of v and w is

$$v \times w = \langle v_2 w_3 - w_3 v_2, \; -(v_1 w_3 - v_3 w_1), \; v_1 w_2 - v_2 w_1 \rangle$$

in $\mathbb{R}^3$. Then

$$\langle u | v \times w \rangle = u_1 (v_2 w_3 - w_3 v_2) - u_2 (v_1 w_3 - w_3 v_1) + u_3 (v_1 w_2 - w_2 v_1)$$

$$= u_1 \cdot \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} + u_2 (-1)^{1+2} \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 (-1)^{1+3} \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

$$= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

by the cofactor formula on row 1 for 3x3 matrices.

If P is the parallelepiped with edge vectors u, v, w then

$$|\langle u | v \times w \rangle| = \left| \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \right| = Vol(P).$$

$u \times v$ is perpendicular to both u and v

Since the determinant of a matrix with two equal rows is 0 then

$$\langle u | u \times v \rangle = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = 0$$

$$\langle v | u \times v \rangle = \det \begin{pmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = 0$$

Topic 3 Example 5 Find a vector perpendicular to both $u = \langle 1, 1, 1 \rangle$ and $v = \langle 1, -1, -2 \rangle$.

The vector

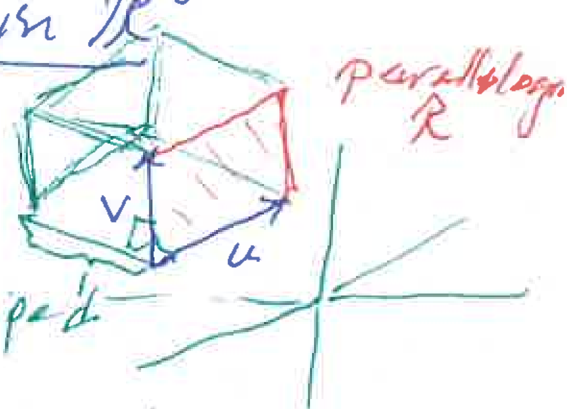
$$u \times v = \langle 1 \cdot (-2) - (-1) \cdot 1, -(1 \cdot (-2) - 1 \cdot 1), 1 \cdot (-1) - 1 \cdot 1 \rangle$$

$$= \langle -2 + 1, -(-2-1), -1-1 \rangle = \langle -1, 3, -2 \rangle$$

is perpendicular to both u and v. $\square$

Area of a parallelogram in $\mathbb{R}^3$

Let
$u = \langle u_1, u_2, u_3 \rangle$
$v = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$



Let
$\hat{n} = \langle n_1, n_2, n_3 \rangle$ be perpendicular to u and v with length 1

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A good formula for $\hat{n}$ is
$$\hat{n} = \frac{1}{\|u\times v\|} u\times v$$
If R is the parallelogram with adjacent edges u and v, and P is the parallelogram with adjacent edges $\hat{n}$, u and v.

Then
1. $\mathrm{Area}(R) = \mathrm{Vol}(P) = |\langle \hat{n} | u\times v\rangle|$
$$= \left|\left\langle \frac{1}{\|u\times v\|} u\times v \,\middle|\, u\times v\right\rangle\right|$$
$$= \frac{1}{\|u\times v\|}\|u\times v\|^2 = \|u\times v\|.$$

<u>Topic 3 Example 6</u>

$u = \vec{PQ} = Q - P$
$= (1,1,1)$
$v = \vec{PR} = R - P$
$= (1,-1,-2)$.

Then $u\times v = (1\cdot(-2)-(-1)\cdot 1, -(1\cdot(-2)-1\cdot 1), 1\cdot(-1)-1\cdot 1)$
$= (-2+1, -(-3), -2) = (-1, 3, -2)$

and $\|u\times v\| = \sqrt{(-1)^2+3^2+(-2)^2} = \sqrt{1+9+4} = \sqrt{14}$.

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Let T be the triangle with vertices P, Q, R and B the parallelogram with adjacent edges $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Then

$$\text{Area}(T) = \frac{1}{2}\text{Area}(B) = \frac{1}{2}\|u \times v\|$$

$$= \frac{1}{2}\sqrt{14}.$$

3d space time

Let $a = \langle a_1, a_2, a_3 \rangle$

$b = \langle b_1, b_2, b_3 \rangle$.

The cross product of a and b is

$$a \times b = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$$

3d-space time is

$$\mathbb{D} = \{ t + xi + yj + zk \mid t, x, y, z \in \mathbb{R} \}$$

with

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1,$$
$$ij = k, \quad jk = i, \quad ki = j,$$
$$ji = -k, \quad kj = -i, \quad ik = -j.$$

1d space time is $\mathbb{C} = \{ t + xi \mid x \in \mathbb{R} \}$ with $i^2 = -1$.

3d space is contained in 3d-space time

$$\mathbb{D} = \{ t + xi + yj + zk \mid t, x, y, z \in \mathbb{R} \}$$
$$\supseteq$$
$$\mathbb{R}^3 = \{ xi + yj + zk \mid x, y, z \in \mathbb{R} \}$$

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Linear Algebra

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Let

$$a = a_1 i + a_2 j + a_3 k$$
$$b = b_1 i + b_2 j + b_3 k$$

be in 3d space

Then

$$ab = (a_1 i + a_2 j + a_3 k)\cdot(b_1 i + b_2 j + b_3 k)$$

$$= a_1 b_1 i^2 + a_2 b_2 j^2 + a_3 b_3 k^2$$
$$+ a_1 b_2 ij + a_1 b_3 ik$$
$$+ a_2 b_1 ji + a_2 b_3 jk$$
$$+ a_3 b_1 ki + a_3 b_2 kj$$

$$= -a_1 b_1 - a_2 b_2 - a_3 b_3$$
$$+ a_1 b_2 k - a_1 b_3 j$$
$$- a_2 b_1 k + a_2 b_3 i$$
$$+ a_3 b_1 j - a_3 b_2 i$$

$$= -\langle a | b \rangle + a \times b.$$

So $a \times b$ arises naturally from multiplication in 3d space-time.