14.08.2013
Linear Algebra ①

## What is a field?

In English: A *field* is a number system with addition, subtraction, multiplication and division.

In Math: A field $\mathbb{F}$ is... SEE THE PROOF MACHINE NOTES.

## Examples of fields

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{C} = \{t + xi \mid t, x \in \mathbb{R}\}$ with $i^2 = -1$.

$\mathbb{F}_{11} = \{0, 1, \ldots, 10\}$ with $11 = 0$.

In $\mathbb{F}_{11}$,

$7^{-1} = 8$ since $8 \cdot 7 = 56 = 5 \cdot 11 + 1 = 5 \cdot 0 + 1$
$= 0 + 1 = 1$.

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## What is a ~~..~~ $\mathbb{F}$-vector space

In English: An $\mathbb{F}$-vector space is a vector system with addition and scalar multiplication.

In Math: An <u>$\mathbb{F}$-vector space</u> is a set $V$ with two functions

$$\begin{array}{ccc} V \times V \to V & \text{and} & \mathbb{F} \times V \to V \\ (v_1, v_2) \mapsto v_1 + v_2 & & (c, v) \mapsto cv \\ \text{addition} & & \text{scalar multiplication} \end{array}$$

such that

A WHOLE BUNCH OF CONDITIONS HOLD.

## Examples of $\mathbb{F}$-vector spaces

$M_{m \times n}(\mathbb{F}) = \{ m \times n \text{ matrices with entries in } \mathbb{F} \}$ with addition and scalar multiplication specified by ...

$\mathbb{F}[x]_{\leq k} = \{ a_0 + a_1 x + \cdots + a_k x^k \mid a_1, \ldots, a_k \in \mathbb{F} \}$ with addition and scalar multiplication specified by...

Let $S$ be a set 14.08.2023 Linear Algebra 3

$\mathcal{F}(S;\mathbb{F}) = \{\text{functions } f: S \to \mathbb{F}\}$

with addition and scalar multiplication given by ...

$\mathbb{F}^n = M_{n \times 1}(\mathbb{F})$

## $\mathbb{F}$-subspaces

Let $V$ be an $\mathbb{F}$-vector space.

An $\mathbb{F}$-subspace of $V$ is a subset $W \subseteq V$ such that

(1) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$.

(2) If $w \in W$ and $c \in \mathbb{F}$ then $cw \in W$

(3) $0 \in W$.

Topic 4 Example 7 Show that

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$$

is an $\mathbb{R}$-subspace of $\mathbb{R}^3$.

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To show: (a) If $w_1, w_2 \in S$ then $w_1 + w_2 \in S$.
(b) If $w \in S$ and $c \in \mathbb{R}$ then $cw \in S$
(c) $0 \in S$.

(a) Assume $w_1, w_2 \in S$.
To show: $w_1 + w_2 \in S$.
Write $w_1 = \langle a, b, c \rangle$ and $w_2 = \langle x, y, z \rangle$.
Since $w_1 \in S$ then $a+b+c=0$. (*)
Since $w_2 \in S$ then $x+y+z=0$. (**)
$w_1 + w_2 = \langle a,b,c \rangle + \langle x,y,z \rangle = \langle a+x, b+y, c+z \rangle$
To show: $(a+x)+(b+y)+(c+z)=0$.
Since $(a+x)+(b+y)+(c+z) = (a+b+c)+(x+y+z)$
in $\mathbb{R}$, and
$a+b+c=0$ and $x+y+z=0$ (by (*) and (**)),
then
$(a+x)+(b+y)+(c+z)=0$.
So $w_1 + w_2 = \langle a+x, b+y, c+z \rangle \in S$
(b) Assume $w \in S$ and $c \in \mathbb{R}$
To show: $cw \in S$.

Since $w \in S$ then there are $x, y, z \in \mathbb{R}$ such that $\langle x, y, z \rangle = w$ and

$$x + y + z = 0. \qquad (***)$$

Then $cw = c\langle x, y, z\rangle = \langle cx, cy, cz\rangle$.

To show: $cx + cy + cz = 0$.

Since $cx + cy + cz = c(x + y + z)$ in $\mathbb{R}$

and $x + y + z = 0$ (by $(***)$)

then

$$cx + cy + cz = 0.$$

So $cw = \langle cx, cy, cz\rangle \in S$.

(4) To show: $0 \in S$.

$0 \in \mathbb{R}^3$ is $0 = \langle 0, 0, 0\rangle$.

To show: $0 + 0 + 0 = 0$.

So $0 + 0 + 0 = 0$ in $\mathbb{R}$ then $0 = \langle 0, 0, 0\rangle \in S$

So $S$ is a subspace of $\mathbb{R}^3$.

<u>Topic 4 Example 8</u> Is

$$L = \{\langle x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$$ a

subspace of $\mathbb{R}^2$?

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Since $O = \langle 0, 0 \rangle \in \mathbb{R}^2$
does not satisfy the condition $y = 2x + 1$
then $O \notin L$.
So $L$ is not a subspace of $\mathbb{R}^2$