Theorem Let $A \in M_{s\times t}(\mathbb{R})$. Then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = t$$

Equivalently,

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = t$$

Examples 33 Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in M_{2\times 3}(\mathbb{F}_2)$.

Recall $\mathbb{F}_2 = \{0, 1\}$ with $0+0=0$, $1+0=1$, $0+1=1$, $1+1=0$, and

$0\cdot 0=0$, $0\cdot 1=0$
$1\cdot 0=0$, $1\cdot 1=1$. Then

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

So

rowspace(A) has basis $\{(1\ 0\ 1), (0\ 1\ 1)\}$

colspace(A) has basis $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

Since

$$\begin{matrix} c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{matrix} \quad \text{has} \quad \begin{matrix} c_1 = -c_3 \\ c_2 = -c_3 \\ c_3 = c_3 \end{matrix} \quad \text{then}$$

solutionspace(A) has basis $\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Example 3D Let

$$W = \{ |a+b, 0, a-b, 0\rangle \mid a, b \in \mathbb{R} \}.$$

Then

$$W = \{ a|1,0,1,0\rangle + b|1,0,-1,0\rangle \mid a, b \in \mathbb{R} \}$$
$$= \mathrm{span}\{ |1,0,1,0\rangle, |1,0,-1,0\rangle \}$$

Then $S = \{ |1,0,1,0\rangle, |1,0,-1,0\rangle \}$ is a basis of W.

$T = \{ |1,0,0,0\rangle, |0,0,1,0\rangle$ is also a basis for W since if

$v_1 = |1,0,1,0\rangle$ and $w_1 = |1,0,0,0\rangle$
$v_2 = |1,0,-1,0\rangle$ $\quad\quad w_2 = |0,0,1,0\rangle$ then

$v_1 = w_1 + w_2$ and $w_1 = \frac{1}{2}v_1 + \frac{1}{2}v_2$
$v_2 = w_1 - w_2$ $\quad\quad w_2 = \frac{1}{2}v_1 - \frac{1}{2}v_2$.

Then

$$C = \{ |1,0,1,0\rangle, |1,0,-1,0\rangle, |0,1,0,0\rangle, |0,0,0,1\rangle \}$$
$$= S \cup \{ |0,1,0,0\rangle, |0,0,0,1\rangle$$

is a basis of $\mathbb{R}^4$.

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Favourite vector spaces and favourite bases

(1) $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots a_n \in \mathbb{R}\} = M_{n\times 1}(\mathbb{R})$

has basis

$\{e_1, e_2, \ldots, e_n\}$ where $e_i = (0, \ldots 0, \overset{i^{\text{th}}}{1}, 0, \ldots 0)$

with 1 in the i^{th} entry and 0 elsewhere

(2) $M_{m\times n}(\mathbb{R})$ has basis

$\left\{ E_{ij} \;\middle|\; \begin{matrix} i \in \{1, \ldots, m\} \\ j \in \{1, \ldots, n\} \end{matrix} \right\}$ where $E_{ij} = \begin{array}{c} \qquad j^{\text{th}} \\ \begin{pmatrix} & 1 & 0 & \end{pmatrix} \end{array}$

with 1 in the (i,j) entry and 0 elsewhere

(3) $P_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$

has basis

$\{1, x, x^2, \ldots, x^n\}$.

(4) The vector space of all polynomials

$\mathbb{R}[x]$ has basis $\{1, x, x^2, x^3, \ldots\}$.