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Linear Algebra (1)
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Linear transformations

Let V be an $\mathbb{R}$-vector space.
Let W be a $\mathbb{R}$-vector space.

An $\mathbb{R}$-linear transformation from V to W is a function $T: V \longrightarrow W$ such that

(a) If $v_1, v_2 \in V$ then $T(v_1+v_2) = T(v_1)+T(v_2)$.

(b) If $v \in V$ and $c \in \mathbb{R}$ then $T(cv) = cT(v)$.

The kernel of T and image of T are

$$\ker(T) = \{v \in V \mid T(v) = 0\}$$
$$\mathrm{im}(T) = \{T(v) \mid v \in V\}$$

Let $\mathcal{B}$ be a basis of V, $\mathcal{B} = \{v_1, \ldots, v_s\}$
C be a basis of W, $C = \{w_1, \ldots, w_t\}$.

The matrix of T with respect to the bases $\mathcal{B}$ and C is

$$T_{C\mathcal{B}} = (a_{ij}) \in M_{t \times s}(\mathbb{R})$$

given by

$$T(v_1) = a_{11} w_1 + \cdots + a_{t1} w_t$$

$$T(v_2) = a_{12} w_1 + \cdots + a_{t2} w_t$$

$$\vdots$$

$$T(v_s) = a_{1s} w_1 + \cdots + w_{ts} w_t.$$

<u>Examples 2 and 14</u> Let $T: P_3 \to P_2$ be given by

$$T(p) = \frac{dp}{dx}.$$

P_3 has basis, $\mathcal{B} = \{1, x, x^2, x^3\}$

P_2 has basis, $C = \{1, x, x^2\}$.

$$T(1) = \frac{d1}{dx} = 0 = 0 + 0x + 0x^2$$

$$T(x) = \frac{dx}{dx} = 1 = 1 + 0x + 0x^2$$

$$T(x^2) = \frac{dx^2}{dx} = 2x = 0 + 2x + 0x^2$$

$$T(x^3) = \frac{dx^3}{dx} = 3x^2 = 0 + 0x + 3x^2.$$

The matrix of T with respect to the bases $\mathcal{B}$ and C is

$$[T]_{CB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Then

$$\ker(T) = \{a_0 + 0x + 0x^2 + 0x^3 \mid a_0 \in \mathbb{R}\}$$
$$= \mathrm{span}\{1\}.$$

$$\mathrm{im}(T) = \{c_0 + c_1x + c_2x^2 \mid c_0, c_1, c_2 \in \mathbb{R}\}$$ since

$$\frac{d}{dx}\left(39 + c_0x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3\right) = c_0 + c_1x + c_2x^2.$$

Then

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = 1 + 3 = 4$$

and $4 = \dim(\mathrm{source}(T))$.

<u>Example 11</u> Let $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(Q) = Q^t.$$

The matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_{2\times 2}(\mathbb{R})$ (the source) and the basis

$$B = \{E_{11}, E_{12}, E_{21}, E_{22}\} \text{ of } M_{2\times 2}(\mathbb{R})$$

(the target) is

$$[T]_{BB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T(E_{11}) = E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{12}) = E_{21} = 0E_{11} + 0E_{12} + 1E_{21} + 0E_{22}$$

$$T(E_{21}) = E_{12} = 0E_{11} + 1E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{22}) = E_{22} = 0E_{11} + 0E_{12} + 0E_{21} + 1 \cdot E_{22}.$$

This has

$$\ker(T) = 0 \quad \text{and} \quad \mathrm{im}(T) = M_{2\times 2}(\mathbb{R}).$$

So

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = 0 + 4 = 4$$

and $4 = \dim(\mathrm{source}(T))$.

Note that, since $(Q^t)^t = Q$ then

$$T^{-1} = T.$$

Thus T is an invertible (reversible) linear transformation.

<u>Example 3</u> Let $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be the linear transformation given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Let $v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$T(v_1 + v_2) = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \quad \text{and}$$

$$T(v_1) + T(v_2) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 = 0.$$

So T does not satisfy the condition

if $v_1, v_2 \in M_{2\times 2}(\mathbb{R})$ then $T(v_1+v_2) = T(v_1) + T(v_2)$

So T is <u>not</u> a linear transformation.

<u>Example 12</u> Let T be the linear transformation

$T: P_2 \to P_1$ given by

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_2) + a_0 x.$$

With respect to the bases

$B = \{1, x, x^2\}$ of P_2 (the source) and

$C = \{1, x\}$ of P_1 (the target)

the matrix of T is

$$[T]_{CB} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ since } \begin{array}{l} T(1) = 1 + 1 \cdot x \\ T(x) = 0 + 0x \\ T(x^2) = 1 + 0x. \end{array}$$

With respect to the basis

$B = \{1, x, x^2\}$ of P_2 (the source) and

$D = \{2, 3x\}$ of P_1 (the target)

the matrix of T is

$$[T]_{DB} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \text{ since } \begin{array}{l} T(1) = \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 3x \\ T(x) = 0 \cdot 2 + 0 \cdot 3x \\ T(x^2) = \frac{1}{2} \cdot 2 + 0 \cdot 3x. \end{array}$$

Then

$\ker(T) = \{a_1 x \mid a_1 \in \mathbb{R}\} = \text{span}\{x\}$ and

$\text{im}(T) = \{c_0 + c_1 x \mid c_0, c_1 \in \mathbb{R}\} = P_1.$

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So $\dim(\ker(H)) + \dim(\operatorname{im}(H)) = 1 + 2 = 3$

and $3 = \dim(P_2) = \dim(\operatorname{source}(H))$.

Example 34 Topic 4

Let $v = \langle 1,5\rangle \in \mathbb{R}^2$.

The coordinate vector of v with respect to the basis $B = \{\langle 1,0\rangle, \langle 0,1\rangle\}$ is

$$[v]_B = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ since } v = \langle 1,5\rangle = 1\cdot\langle 1,0\rangle + 5\cdot\langle 0,1\rangle$$

The coordinate vector of v with respect to the basis $C = \{\langle 2,1\rangle, \langle -1,1\rangle\}$ is

$$[v]_C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

since $\langle 1,5\rangle = \cancel{3\cdot\langle 2,1\rangle + 2\cdot\langle -1,1\rangle}$

$= 2\cdot\langle 2,1\rangle + 3\langle -1,1\rangle$

Example 35 Topic 4

The coordinate vector of $p = 2 + 7x - 9x^2$ with respect to the basis

$B = \{2, \tfrac{1}{2}x, -3x^2\}$ of P_2 is

$[p]_B = \begin{pmatrix} 1 \\ 14 \\ 3 \end{pmatrix}$

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Since

$$2 + 7x - 9x^2 = 1 \cdot 2 + 14 \cdot \tfrac{1}{2}x + 3 \cdot (-3x^2)$$